A SCHWARZ LEMMA FOR HARMONIC MAPPINGS BETWEEN THE UNIT BALLS IN REAL EUCLIDEAN SPACES

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Abstract. In this paper we prove a Schwarz lemma for harmonic mappings between the unit balls in real Euclidean spaces. Roughly speaking, our result says that under a harmonic mapping between the unit balls in real Euclidean spaces, the image of a smaller ball centered at origin can be controlled. This extends the related result proved by Chen in complex plane.

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1. Introduction

Let n be a positive integer greater than 1. $\mathbb{R}^n$ is the real space of dimension n. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $|x| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$. Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball of $\mathbb{R}^n$. The unit sphere, the boundary of $\mathbb{B}^n$ is denoted by S; normalized surface-area measure on S is denoted by σ (so that $\sigma(S) = 1$). Let S^+ denote the northern hemisphere $\{x = (x_1, \cdots, x_n) \in S : x_n > 0\}$ and let S^- denote the southern hemisphere $\{x = (x_1, \cdots, x_n) \in S : x_n < 0\}$. $N = (0, \cdots, 0, 1)$ denotes the north pole of S. $B_r^n = \{x \in \mathbb{R}^n : |x| < r\}$ is the open ball centered at origin of radius r; its closure is the closed ball $\overline{B_r^n}$.

Let m be a positive integer with $m \geq 1$. A mapping $F = (F_1, \cdots, F_m, F_{m+1})$ from $\mathbb{B}^n$ into $\mathbb{B}^{m+1}$ is harmonic on $\mathbb{B}^n$ if and only if for $k = 1, \cdots, m, m+1$, F_k is twice continuously differentiable and $\Delta F_k \equiv 0$, where $\Delta = D_1^2 + \cdots + D_n^2$ and D_j^2 denotes the second partial derivative with respect to the j^{th} coordinate variable x_j. By $\Omega_{n,m+1}$, we denote the class of all harmonic mappings F from $\mathbb{B}^n$ into $\mathbb{B}^{m+1}$.

Let $\mathfrak{B}^n$ be the unit ball in the complex space $\mathbb{C}^n$. Denote the ball $\{z \in \mathbb{C}^n : |z| < r\}$ by $\mathfrak{B}_r^n$; its closure is the closed ball $\overline{\mathfrak{B}_r^n}$. For a holomorphic mapping f from $\mathfrak{B}^n$ into $\mathfrak{B}^m$, the classical Schwarz lemma [1] says that if $f(0) = 0$, then

$$|f(z)| \leq |z| \tag{1.1}$$

holds for $z \in \mathfrak{B}^n$. For $0 < r < 1$, (1.1) may be written in the following form:

$$f(\overline{\mathfrak{B}_r^n}) \subset \overline{\mathfrak{B}_r^m}.$$

So the classical Schwarz lemma can be regarded as considering the region of $f(\overline{\mathfrak{B}_r^n})$. If $f(0) \neq 0$, then what the region of $f(\overline{\mathfrak{B}_r^n})$ is. It seems that there is not much of research in the literature. However, the same problem also exists in harmonic mappings. The work in the following by Chen [2] seems to be the first result of this kind of study for harmonic mappings in the complex plane.

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. Denote the disk $\{z \in \mathbb{C} : |z| < r\}$ by D_r; its closure is the closed disk $\overline{D}_r$. For $0 < r < 1$ and $0 \leq \rho < 1$, Chen [2] constructed a closed domain $E_{r,\rho}$ and proved that

Theorem A. *Let $0 \leq \rho < 1$, $\alpha \in \mathbb{R}$ and $0 < r < 1$ be given. For every complex-valued harmonic function F on $\mathbb{D}$ such that $F(\mathbb{D}) \subset \mathbb{D}$, if $F(0) = \rho e^{i\alpha}$, then*

$$F(\overline{D}_r) \subset e^{i\alpha} E_{r,\rho}, \tag{1.2}$$

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which is sharp.

Note that the function F in the above theorem can be seen as $F \in \Omega_{2,2}$. So (1.2) can be regarded as considering the region of $F(\overline{B_r^2})$ when $F \in \Omega_{2,2}$ regardless of $F(0)=0$ or $F(0) \neq 0$. In [2], the most important theorem for the proof of Theorem A is the theorem as follow, which is the motivation for our study of the extremal mapping. The mappings $U_{a,b,r}$ and $F_{a,b,r}$ in the following theorem are defined in [2].

Theorem B. *Let $F = U + iV$ be a harmonic mapping such that $F(\mathbb{D}) \subset \mathbb{D}$ and $F(0) = a + bi$. Then for $0 < r < 1$ and $0 \leq \theta \leq 2\pi$,*

$$U(re^{i\theta}) \leq U_{a,b,r}(ri)$$

with equality at some point $re^{i\theta}$ if and only if $F(z) = F_{a,b,r}(e^{i(\pi/2-\theta)}z)$. Furthermore, $U(z) < U_{a,b,r}(ri)$ for $|z| < r$.

A classical Schwarz lemma for complex-valued harmonic function on $\mathbb{B}^n$ [3] says that

Theorem C. *Suppose that F is a complex-valued harmonic function on $\mathbb{B}^n$, $|F| < 1$ on $\mathbb{B}^n$, and $F(0) = 0$. Then*

$$|F(x)| \leq U(|x|N) \tag{1.3}$$

holds for every $x \in \mathbb{B}^n$, where U is the Poisson integral of the function that equals 1 on S^+ and -1 on S^-. Equality holds for some nonzero $x \in \mathbb{B}^n$ if and only if $F = \lambda(U \circ A)$ where λ is a complex constant of modulus 1 and A is an orthogonal transformation.

Especially, when $n = 2$ in the above theorem, it is known [4] that

$$|F(x)| \leq \frac{4}{\pi} \arctan |x|$$

holds for every $x \in \mathbb{B}^2$.

From Theorem C, for $0 < r < 1$, (1.3) may be written in the following form:

$$F(\overline{B_r^n}) \subset \overline{D}_{U(rN)}, \tag{1.4}$$

where $\overline{D}_{U(rN)} = \{z \in \mathbb{C} : |z| \leq U(rN)\}$.

Note that the function F in the above Theorem C can be seen as $F \in \Omega_{n,2}$. So (1.4) can be regarded as considering the region of $F(\overline{B_r^n})$ when $F \in \Omega_{n,2}$ with $F(0) = 0$. It is natural to consider that if $F \in \Omega_{n,2}$ with $F(0) \neq 0$, then what the region of $F(\overline{B_r^n})$ is. Furthermore, we want to know that for the general $F \in \Omega_{n,m+1}$, what the estimate corresponding to (1.4) is when $F(0) = 0$ or $F(0) \neq 0$. This problem will be resolved in this paper. When $F(0) \neq 0$, this problem is serious because the composition $f \circ F$ of a möbius transformation f and a harmonic mapping F does not need to be harmonic.

In this paper, inspired by the method of the proof of Theorem B in [2], we obtain the following Theorem 1, which is very important in this paper. (1.5) is the estimate corresponding to (1.3) without the assumption $F(0) = 0$. Especially, when $F(0) = 0$, we have Corollary 1, which is coincident with Theorem C when $m + 1 = 2$. Note that in the following theorem, $F_{(a,b)Q_e,r}$ is defined as (3.23).

Theorem 1. *Let $F(x)$ be a harmonic mapping such that $F(\mathbb{B}^n) \subset \mathbb{B}^{m+1}$ and $F(0) = (a, b)$, where $a \in \mathbb{R}^m$ and $b \in \mathbb{R}$. Let e be a unit vector in $\mathbb{R}^{m+1}$, $e_0 = (1, 0, \cdots, 0) \in \mathbb{R}^{m+1}$ and Q_e be an orthogonal matrix such that $eQ_e = e_0$. Then, for $0 < r < 1$ and $\omega \in S$,*

$$\langle F(r\omega), e\rangle \leq \langle F_{(a,b)Q_e,r}(rN), e_0\rangle \tag{1.5}$$

with equality at some point $r\omega$ if and only if $F(x) = F_{(a,b)Q_e,r}(xA)Q_e^{-1}$, where A is an orthogonal matrix such that $\omega A = N$ and Q_e^{-1} is the inverse matrix of Q_e. Furthermore, $\langle F(x), e\rangle < \langle F_{(a,b)Q_e,r}(rN), e_0\rangle$ for $|x| < r$.

Corollary 1. *Let $F(x)$ be a harmonic mapping such that $F(\mathbb{B}^n) \subset \mathbb{B}^{m+1}$ and $F(0) = 0$. Then*

$$|F(x)| \leq U(|x|N)$$

for every $x \in \mathbb{B}^n$, where U is the Poisson integral of the function that equals 1 on S^+ and -1 on S^-. Equality holds for some nonzero $x_0 \in \mathbb{B}^n$ if and only if $F(x) = U(xA)e$, where A is an orthogonal matrix such that $x_0 A = |x_0|N$, e is a unit vector in $\mathbb{R}^{m+1}$.

From Theorem 1, we deduce the following theorem, which is called a harmonic Schwarz lemma for $F \in \Omega_{n,m+1}$ and which resolves the problem we want to know above. Theorem 2 extends Theorem A and is coincident with Theorem A when $n = m + 1 = 2$. Note that in the following theorem, $F_{(a,b)Q_e,r}$ is defined as (3.23).

Theorem 2. *Let $F(x)$ be a harmonic mapping such that $F(\mathbb{B}^n) \subset \mathbb{B}^{m+1}$ and $F(0) = (a, b)$, where $a \in \mathbb{R}^m$ and $b \in \mathbb{R}$. Let $0 < r < 1$. Then*

$$F(\overline{B^n_r}) \subset E_{r,(a,b)}, \tag{1.6}$$

where

$$E_{r,(a,b)} = \bigcap_{e \in \mathbb{R}^{m+1}, |e|=1} R_e,$$

$$R_e = \{x \in \mathbb{R}^{m+1} : \langle x, e\rangle \leq \langle F_{(a,b)Q_e,r}(rN), e_0\rangle\},$$

$e_0 = (1, 0, \cdots, 0) \in \mathbb{R}^{m+1}$ and Q_e be an orthogonal matrix such that $eQ_e = e_0$.

Note that $E_{r,(a,b)}$ in Theorem 2 is a region enveloped by all the hyperplanes

$$P_e = \{x \in \mathbb{R}^{m+1} : \langle x, e\rangle = \langle F_{(a,b)Q_e,r}(rN), e_0\rangle\},$$

which is the boundary of R_e. By Theorem 1, it is obviously that the region $E_{r,(a,b)}$ is sharp. This means that under $F \in \Omega_{n,m+1}$, the image of a small ball centered at origin of radius r can be controlled.

In section 2, we will give two main lemmas. The proofs of the lemmas will be given in section 4. In section 3, the main results of this paper and the proofs will be given.

2. The main lemmas

In this section, we will introduce two main lemmas, which are important for the proof of Theorem 3 and which extend the related lemmas proved by Chen in [2]. Lemma 1 constructs a bijection (R, I) from $\mathbb{R}^m \times \mathbb{R}^+$ onto the upper half ball $\{(a, b) : a \in \mathbb{R}^m, b \in \mathbb{R}, |a|^2 + b^2 < 1,\ b > 0\}$, which will be used to construct $u_{a,b,r}$ in Theorem 3 for the case that $b > 0$. Lemma 2 constructs a bijection $\mathcal{R}$ from $\mathbb{R}^m$ onto the ball $\{a : a \in \mathbb{R}^m, |a| < 1\}$, which will be used to construct $u_{a,b,r}$ in Theorem 3 for the case that $b = 0$. Now we give the two main lemmas. The proofs of Lemma 1 and Lemma 2 will be given in section 4.

For $0 < r < 1$, $\mu > 0$, $\lambda \in \mathbb{R}^m$, and $l = (1, 0, \cdots, 0) \in \mathbb{R}^m$, define

$$A_{r,\lambda,\mu}(\omega) = \frac{1}{\mu}\left(\frac{1}{|rN - \omega|^n} l - \lambda\right), \qquad \omega \in S, \tag{2.1}$$

and

$$R(r, \lambda, \mu) = \int_S \frac{A_{r,\lambda,\mu}(\omega)}{\sqrt{1 + |A_{r,\lambda,\mu}(\omega)|^2}}\, d\sigma, \quad I(r, \lambda, \mu) = \int_S \frac{1}{\sqrt{1 + |A_{r,\lambda,\mu}(\omega)|^2}}\, d\sigma. \tag{2.2}$$

The idea of the conformation of $A_{r,\lambda,\mu}(\omega)$, $R(r, \lambda, \mu)$ and $I(r, \lambda, \mu)$ originates from (3.5) and (3.10).

Lemma 1. *Let $0 < r < 1$ be fixed. Then, there exist a unique pair of continuous mappings $\lambda = \lambda(r, a, b) \in \mathbb{R}^m$ and $\mu = \mu(r, a, b) > 0$, defined on the upper half ball $\{(a, b) : a \in \mathbb{R}^m, b \in \mathbb{R}, |a|^2 + b^2 < 1,\ b > 0\}$, such that $R(r, \lambda(r, a, b), \mu(r, a, b)) = a$ and $I(r, \lambda(r, a, b), \mu(r, a, b)) = b$ for any point (a, b) in the half ball.*

For $0 < r < 1$, $\lambda \in \mathbb{R}^m$, and $l = (1, 0, \cdots, 0) \in \mathbb{R}^m$, define

$$\mathcal{A}_{r,\lambda}(\omega) = \frac{1}{|rN - \omega|^n} l - \lambda, \qquad \omega \in S, \tag{2.3}$$

and

$$\mathcal{R}(r, \lambda) = \int_S \frac{\mathcal{A}_{r,\lambda}(\omega)}{|\mathcal{A}_{r,\lambda}(\omega)|} \, d\sigma. \tag{2.4}$$

The idea of the conformation of $\mathcal{A}_{r,\lambda}(\omega)$ and $\mathcal{R}(r, \lambda)$ originates from (3.15). Note that $\mathcal{R}(r, \lambda)$ is well defined, since $|\mathcal{A}_{r,\lambda}(\omega)| \neq 0$ except for a zero measure set of ω at most.

Lemma 2. *Let $0 < r < 1$ be fixed. Then, there exist a unique continuous mapping $\lambda = \lambda(r, a) \in \mathbb{R}^m$, defined on $\{a : a \in \mathbb{R}^m, |a| < 1\}$, such that $\mathcal{R}(r, \lambda(r, a)) = a$ for any point a.*

3. The main results

Let $a \in \mathbb{R}^m$, $b \in \mathbb{R}$ and $0 \leq b < 1$, $|a|^2 + b^2 < 1$. Let $\mathcal{U}_{a,b}$ denote the class of mappings $u \in (L^\infty(S))^m$ satisfying the following conditions:

$$\|u\|_\infty \leq 1, \quad \int_S u(\omega) d\sigma = a, \quad \int_S \sqrt{1 - |u(\omega)|^2} d\sigma \geq b. \tag{3.1}$$

Every function $u \in (L^\infty(S))^m$ defines a harmonic mapping

$$U(x) = \int_S \frac{1 - |x|^2}{|x - \omega|^n} u(\omega) d\sigma \quad \text{for} \quad x \in \mathbb{B}^n.$$

Let $0 < r < 1$, $l = (1, 0, \cdots, 0) \in \mathbb{R}^m$ and define a functional L_r on $(L^\infty(S))^m$ by

$$L_r(u) = \langle U(rN), l \rangle = \int_S \frac{1 - r^2}{|rN - \omega|^n} \langle u(\omega), l \rangle d\sigma. \tag{3.2}$$

Obviously, $\mathcal{U}_{a,b}$ is a closed set, and L_r is a continuous functional on $\mathcal{U}_{a,b}$. Then there exists a extremal mapping such that L_r attains its maximum on $\mathcal{U}_{a,b}$ at the extremal mapping. We will claim in the following theorem that the extremal mapping is unique. In the proof of the following theorem, we will construct a mapping u_0 first and then prove that u_0 is the unique extremal mapping, which will be denoted by $u_{a,b,r}$.

Theorem 3. *For any a, b and r satisfying the above conditions, there exists a unique extremal mapping $u_{a,b,r} \in \mathcal{U}_{a,b}$ such that L_r attains its maximum on $\mathcal{U}_{a,b}$ at $u_{a,b,r}$.*

For the proof of Theorem 3, we need Lemma 1, Lemma 2 and the lemma in the following.

Lemma 3. *Let $x, y \in \mathbb{R}^m$, $|x| \leq 1$ and $|y| < 1$. Then*

$$\sqrt{1 - |y|^2} - \sqrt{1 - |x|^2} = \frac{\langle x - y, y \rangle}{\sqrt{1 - |y|^2}} + \frac{|x - y|^2 (1 - |\tilde{y}|^2) + |\langle x - y, \tilde{y} \rangle|^2}{2(1 - |\tilde{y}|^2)^{3/2}} \tag{3.3}$$

holds, where $\tilde{y} = y + \zeta(x - y)$, $0 < \zeta < 1$.

Proof. Let $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_m)$ and $g(x) = \sqrt{1 - |x|^2}$. For $j = 1, \cdots, m$ and $k = 1, \cdots, m$, denote $\frac{\partial g(x)}{\partial x_j}$ by $g_j(x)$ and $\frac{\partial^2 g(x)}{\partial x_j \partial x_k}$ by $g_{jk}(x)$. Then

$$g_j(x) = -\frac{x_j}{\sqrt{1 - |x|^2}},$$

$$\frac{\partial^2 g(x)}{\partial x_j \partial x_k} = -\frac{\delta_{jk}(1 - |x|^2) + x_j x_k}{(1 - |x|^2)^{3/2}},$$

where

$$\delta_{jk} = \begin{cases} 1, & j = k; \\ 0, & j \neq k. \end{cases}$$

Let $\varphi(t) = g(y + t(x - y))$. By Taylor formula, we have

$$\varphi(1) - \varphi(0) = \varphi'(0) + \frac{1}{2}\varphi''(\zeta) \qquad (0 < \zeta < 1). \tag{3.4}$$

Note that

$$\varphi(0) = g(y) = \sqrt{1 - |y|^2}, \qquad \varphi(1) = g(x) = \sqrt{1 - |x|^2},$$

$$\varphi'(t) = \sum_{j=1}^{m} g_j(y + t(x - y)) \cdot (x_j - y_j),$$

$$\varphi'(0) = \sum_{j=1}^{m} g_j(y) \cdot (x_j - y_j) = -\frac{\langle x - y, y\rangle}{\sqrt{1 - |y|^2}},$$

$$\varphi''(t) = \sum_{j,k=1}^{m} g_{jk}(y + t(x - y)) \cdot (x_j - y_j)(x_k - y_k),$$

$$\begin{aligned}
\varphi''(\zeta) &= \sum_{j,k=1}^{m} g_{jk}(\tilde{y}) \cdot (x_j - y_j)(x_k - y_k) \\
&= \sum_{j,k=1}^{m} -\frac{\delta_{jk}(1 - |\tilde{y}|^2) + \tilde{y}_j\tilde{y}_k}{(1 - |\tilde{y}|^2)^{3/2}} \cdot (x_j - y_j)(x_k - y_k) \\
&= -\frac{|x - y|^2(1 - |\tilde{y}|^2) + \sum_{j,k=1}^{m} \tilde{y}_j\tilde{y}_k(x_j - y_j)(x_k - y_k)}{(1 - |\tilde{y}|^2)^{3/2}} \\
&= -\frac{|x - y|^2(1 - |\tilde{y}|^2) + |\langle x - y, \tilde{y}\rangle|^2}{(1 - |\tilde{y}|^2)^{3/2}},
\end{aligned}$$

where $\tilde{y} = y + \zeta(x - y)$. Then by (3.4), (3.3) is proved. $\square$

Now we give the proof of Theorem 3.

Proof of Theorem 3. Let a, b and r be fixed. First assume that $b > 0$. From Lemma 1, we have $\lambda = \lambda(r, a, b)$ and $\mu = \mu(r, a, b) > 0$ such that $R(r, \lambda, \mu) = a$ and $I(r, \lambda, \mu) = b$. For the need of (3.10), let

$$u_0(\omega) = \frac{A_{r,\lambda,\mu}(\omega)}{\sqrt{1 + |A_{r,\lambda,\mu}(\omega)|^2}}, \tag{3.5}$$

where $A_{r,\lambda,\mu}(\omega)$ is defined as (2.1). Then $||u_0||_\infty < 1$ and by (2.2), we know

$$\int_S u_0(\omega)d\sigma = R(r, \lambda, \mu) = a, \quad \int_S \sqrt{1 - |u_0(\omega)|^2}d\sigma = I(r, \lambda, \mu) = b. \tag{3.6}$$

This means that $u_0 \in \mathcal{U}_{a,b}$.

Let $u \in \mathcal{U}_{a,b}$. By (3.1) and (3.6), we have

$$\int_S \langle u_0(\omega) - u(\omega), \lambda\rangle d\sigma = 0, \tag{3.7}$$

$$\mu \int_S (\sqrt{1 - |u_0(\omega)|^2} - \sqrt{1 - |u(\omega)|^2})d\sigma \le 0. \tag{3.8}$$

By Lemma 3, we have

$$\begin{aligned}
&\sqrt{1 - |u_0(\omega)|^2} - \sqrt{1 - |u(\omega)|^2} \\
&= \frac{\langle u(\omega) - u_0(\omega), u_0(\omega)\rangle}{\sqrt{1 - |u_0(\omega)|^2}} + \frac{|u(\omega) - u_0(\omega)|^2(1 - |\tilde{u}(\omega)|^2) + |\langle u(\omega) - u_0(\omega), \tilde{u}(\omega)\rangle|^2}{2(1 - |\tilde{u}(\omega)|^2)^{3/2}},
\end{aligned} \tag{3.9}$$

where $\tilde{u}(\omega) = u_0(\omega) + \zeta(u(\omega) - u_0(\omega))$, $0 < \zeta < 1$. By (3.5) and (2.1), we have

$$\frac{1}{|rN - \omega|^n} l - \lambda - \frac{\mu u_0(\omega)}{\sqrt{1 - |u_0(\omega)|^2}} = 0. \tag{3.10}$$

Then by (3.2) and (3.7)-(3.10), we obtain that

$$\begin{aligned}
\frac{L_r(u_0) - L_r(u)}{1 - r^2} &= \int_S \frac{\langle u_0(\omega) - u(\omega), l\rangle}{|rN - \omega|^n} d\sigma \\
\geq \quad & \int_S \frac{\langle u_0(\omega) - u(\omega), l\rangle}{|rN - \omega|^n} d\sigma - \int_S \langle u_0(\omega) - u(\omega), \lambda\rangle d\sigma \\
& \qquad + \mu \int_S (\sqrt{1 - |u_0(\omega)|^2} - \sqrt{1 - |u(\omega)|^2}) d\sigma \\
= \quad & \int_S \langle u_0(\omega) - u(\omega), \frac{1}{|rN - \omega|^n} l - \lambda - \frac{\mu u_0(\omega)}{\sqrt{1 - |u_0(\omega)|^2}} \rangle d\sigma \\
& \qquad + \mu \int_S \frac{|u(\omega) - u_0(\omega)|^2 (1 - |\tilde{u}(\omega)|^2) + |\langle u(\omega) - u_0(\omega), \tilde{u}(\omega)\rangle|^2}{2(1 - |\tilde{u}(\omega)|^2)^{3/2}} d\sigma \\
= \quad & \mu \int_S \frac{|u(\omega) - u_0(\omega)|^2 (1 - |\tilde{u}(\omega)|^2) + |\langle u(\omega) - u_0(\omega), \tilde{u}(\omega)\rangle|^2}{2(1 - |\tilde{u}(\omega)|^2)^{3/2}} d\sigma.
\end{aligned}$$

Note that

$$\begin{aligned}
||\tilde{u}(\omega)|| &= ||u_0(\omega) + \zeta(u(\omega) - u_0(\omega))|| \\
&= ||u_0(\omega)(1 - \zeta) + \zeta u(\omega)|| \\
&\leq ||u_0(\omega)||(1 - \zeta) + ||u(\omega)||\zeta \\
&< 1 - \zeta + \zeta = 1.
\end{aligned}$$

Thus $L_r(u_0) \geq L_r(u)$ with equality if and only if $u(\omega) = u_0(\omega)$ almost everywhere on S. This shows that $u_0(\omega)$ is the unique extremal mapping, which will be denoted by $u_{a,b,r}(\omega)$.

Next we consider the case that $b = 0$. For the need of (3.18), let

$$u_0(\omega) = \frac{\mathcal{A}_{r,\lambda(r,a)}(\omega)}{|\mathcal{A}_{r,\lambda(r,a)}(\omega)|}, \tag{3.11}$$

where $\lambda(r, a)$ and $\mathcal{A}_{r,\lambda(r,a)}(\omega)$ are defined in Lemma 2. Obviously, $||u_0||_\infty \leq 1$,

$$\int_S \sqrt{1 - |u_0(\omega)|^2} d\sigma = 0, \tag{3.12}$$

and by Lemma 2,

$$\int_S u_0(\omega) d\sigma = \mathcal{R}(r, \lambda(r, a)) = a. \tag{3.13}$$

This means that $u_0 \in \mathcal{U}_{a,0}$.

Let $u \in \mathcal{U}_{a,0}$. By (3.1) and (3.13), we have

$$\int_S \langle u_0(\omega) - u(\omega), \lambda(r, a)\rangle d\sigma = 0. \tag{3.14}$$

By (2.3), we have

$$\frac{1}{|rN - \omega|^n} l - \lambda(r, a) = \mathcal{A}_{r,\lambda(r,a)}(\omega). \tag{3.15}$$

By $||u||_\infty \leq 1$, we have

$$|\langle u(\omega), \mathcal{A}_{r,\lambda(r,a)}(\omega)\rangle| \leq |u(\omega)||\mathcal{A}_{r,\lambda(r,a)}(\omega)| \leq |\mathcal{A}_{r,\lambda(r,a)}(\omega)|, \tag{3.16}$$

and

$$|\mathcal{A}_{r,\lambda(r,a)}(\omega)| = \langle u(\omega), \mathcal{A}_{r,\lambda(r,a)}(\omega)\rangle \quad \text{if and only if} \quad u(\omega) = \frac{\mathcal{A}_{r,\lambda(r,a)}(\omega)}{|\mathcal{A}_{r,\lambda(r,a)}(\omega)|} = u_0(\omega). \tag{3.17}$$

Then by (3.2), (3.11) and (3.14)-(3.17), we obtain that

$$\begin{aligned}
\frac{L_r(u_0) - L_r(u)}{1-r^2} &= \int_S \frac{\langle u_0(\omega) - u(\omega), l\rangle}{|rN-\omega|^n} d\sigma \\
&= \int_S \frac{\langle u_0(\omega) - u(\omega), l\rangle}{|rN-\omega|^n} d\sigma - \int_S \langle u_0(\omega) - u(\omega), \lambda(r,a)\rangle d\sigma \\
&= \int_S \langle u_0(\omega) - u(\omega), \frac{1}{|rN-\omega|^n} l - \lambda(r,a)\rangle d\sigma \\
&= \int_S \langle u_0(\omega) - u(\omega), \mathcal{A}_{r,\lambda(r,a)}(\omega)\rangle d\sigma \\
&= \int_S \langle \frac{\mathcal{A}_{r,\lambda(r,a)}(\omega)}{|\mathcal{A}_{r,\lambda(r,a)}(\omega)|} - u(\omega), \mathcal{A}_{r,\lambda(r,a)}(\omega)\rangle d\sigma \\
&= \int_S (|\mathcal{A}_{r,\lambda(r,a)}(\omega)| - \langle u(\omega), \mathcal{A}_{r,\lambda(r,a)}(\omega)\rangle) d\sigma \geq 0
\end{aligned} \tag{3.18}$$

with equality if and if $u(\omega) = u_0(\omega)$ almost everywhere on S. Thus $L_r(u_0) \geq L_r(u)$ with equality if and if $u(\omega) = u_0(\omega)$ almost everywhere on S. The theorem is proved. □

Let $a \in \mathbb{R}^m$, $b \in \mathbb{R}$, $|a|^2 + b^2 < 1$, and $0 < r < 1$. If $b \geq 0$, $u_{a,b,r}$ has been defined in Theorem 3. Now, define

$$v_{a,b,r}(\omega) = \sqrt{1 - |u_{a,b,r}(\omega)|^2} \quad \text{for} \quad \omega \in S, \tag{3.19}$$

and

$$U_{a,b,r}(x) = \int_S \frac{1-|x|^2}{|x-\omega|^n} u_{a,b,r}(\omega) d\sigma, \tag{3.20}$$

$$V_{a,b,r}(x) = \int_S \frac{1-|x|^2}{|x-\omega|^n} v_{a,b,r}(\omega) d\sigma. \tag{3.21}$$

For $b < 0$, let

$$U_{a,b,r}(x) = U_{a,-b,r}(x), \quad V_{a,b,r}(x) = -V_{a,-b,r}(x). \tag{3.22}$$

Then for any $a \in \mathbb{R}^m$, $b \in \mathbb{R}$ and $|a|^2 + b^2 < 1$, let

$$F_{a,b,r}(x) = (U_{a,b,r}(x), V_{a,b,r}(x)) \quad \text{for} \quad x \in \mathbb{B}^n. \tag{3.23}$$

The harmonic mapping $F_{a,b,r}(x)$ satisfies $F_{a,b,r}(0) = (a,b)$ and $F_{a,b,r}(\mathbb{B}^n) \subset \mathbb{B}^{m+1}$, since we will show that $|U_{a,b,r}(x)|^2 + |V_{a,b,r}(x)|^2 < 1$. By the convexity of the square function,

$$|U_{a,b,r}(x)|^2 + |V_{a,b,r}(x)|^2 \leq \int_S \frac{1-|x|^2}{|x-\omega|^n} (|u_{a,b,r}(\omega)|^2 + v_{a,b,r}^2(\omega)) d\sigma = 1$$

with equality if and only if $u_{a,b,r,1}(\omega), u_{a,b,r,2}(\omega), \cdots, u_{a,b,r,m}(\omega)$ and $v_{a,b,r}(\omega)$ are constants almost everywhere on S, where

$$u_{a,b,r}(\omega) = (u_{a,b,r,1}(\omega), u_{a,b,r,2}(\omega), \cdots, u_{a,b,r,m}(\omega)).$$

However $u_{a,b,r,1}(\omega), u_{a,b,r,2}(\omega), \cdots, u_{a,b,r,m}(\omega)$ are not possiblely constants almost everywhere on S. Thus $|U_{a,b,r}(x)|^2 + |V_{a,b,r}(x)|^2 < 1$.

The mappings $F_{a,b,r}$ are the extremal mappings in the following theorem. Theorem 4 extends Theorem B to $F \in \Omega_{n,m+1}$, and when $n = m+1 = 2$, Theorem 4 is coincident with Theorem B. Note that in the following theorem, $U_{a,b,r}$ is defined as (3.20) and (3.22), $F_{a,b,r}$ is defined as (3.23).

Theorem 4. *Let $F(x) = (U(x), V(x))$ be a harmonic mapping such that $F(\mathbb{B}^n) \subset \mathbb{B}^{m+1}$ and $F(0) = (a, b)$, where $U(x) \in \mathbb{R}^m$, $V(x) \in \mathbb{R}$, $a \in \mathbb{R}^m$ and $b \in \mathbb{R}$. Let $l = (1, 0, \cdots, 0) \in \mathbb{R}^m$. Then, for $0 < r < 1$ and $\omega \in S$,*

$$\langle U(r\omega), l\rangle \le \langle U_{a,b,r}(rN), l\rangle$$

with equality at some point $r\omega$ if and only if $F(x) = F_{a,b,r}(xA)$, where A is an orthogonal matrix such that $\omega A = N$. Further, $\langle U(x), l\rangle < \langle U_{a,b,r}(rN), l\rangle$ for $|x| < r$.

Proof. Step 1: First the case that $r\omega = rN$ will be proved. Let $0 < \tilde{r} < 1$ be fixed. Construct mapping

$$G(x) = F(\tilde{r}x) \quad \text{for} \quad x \in \overline{\mathbb{B}}^n.$$

$G(x)$ is harmonic on $\overline{\mathbb{B}}^n$ and $G(0) = (a, b)$. Let $G(x) = (u(x), v(x))$, where $u(x) \in \mathbb{B}^m$. Then

$$\|u\|_\infty \le 1, \qquad \int_S u(\omega) d\sigma = a,$$

$$\int_S \sqrt{1 - |u(\omega)|^2} d\sigma \ge \int_S |v(\omega)| d\sigma \ge \left| \int_S v(\omega) d\sigma \right| = |b|. \tag{3.24}$$

So by (3.1) we know that $u \in \mathcal{U}_{a,|b|}$, and by Theorem 3 we have

$$\langle u(rN), l\rangle \le \langle U_{a,|b|,r}(rN), l\rangle$$

with equality if and only if $u(\omega) = u_{a,|b|,r}(\omega)$ almost everywhere on S. For $u_{a,|b|,r}(\omega)$, by (3.6) and (3.12) we have

$$\int_S \sqrt{1 - |u_{a,|b|,r}(\omega)|^2} d\sigma = |b|. \tag{3.25}$$

If $u(\omega) = u_{a,|b|,r}(\omega)$ almost everywhere on S, then by (3.20) and (3.22), we have

$$u(x) = U_{a,|b|,r}(x) = U_{a,b,r}(x) \ \text{ for } x \in \mathbb{B}^n;$$

and by (3.19), we have

$$v_{a,|b|,r}(\omega) = \sqrt{1 - |u_{a,|b|,r}(\omega)|^2} = \sqrt{1 - |u(\omega)|^2}. \tag{3.26}$$

Note that by (3.24), (3.25) and (3.26) we have

$$|b| = \int_S v_{a,|b|,r}(\omega) d\sigma \ge \int_S |v(\omega)| d\sigma \ge \left| \int_S v(\omega) d\sigma \right| = |b|.$$

Then

$$v(\omega) = v_{a,|b|,r}(\omega) \quad \text{almost everywhere on} \quad S \quad \text{when} \quad b \ge 0,$$
$$v(\omega) = -v_{a,|b|,r}(\omega) \quad \text{almost everywhere on} \quad S \quad \text{when} \quad b < 0.$$

So

$$v(x) = V_{a,b,r}(x) \ \text{ for } x \in \mathbb{B}^n.$$

For $G(x) = (u(x), v(x))$, it is proved that $\langle u(rN), l\rangle \le \langle U_{a,b,r}(rN), l\rangle$ with equality if and only if $G(x) = F_{a,b,r}(x)$. Now let $\tilde{r} \to 1$. Note that

$$\lim_{\tilde{r} \to 1} G(x) = \lim_{\tilde{r} \to 1} F(\tilde{r}x) = F(x), \quad \lim_{\tilde{r} \to 1} u(rN) = U(rN).$$

Then by the result for $G(x)$, we have $\langle U(rN), l\rangle \le \langle U_{a,b,r}(rN), l\rangle$ with equality if and only if $F(x) = F_{a,b,r}(x)$.

Step 2: Now we prove the case that $r\omega \ne rN$. Construct mapping

$$\tilde{F}(x) = F(xA^{-1}) \quad \text{for} \quad x \in \mathbb{B}^n,$$

where A is an orthogonal matrix such that $r\omega A = rN$ and A^{-1} is the inverse matrix of A. By [3], we know that $\tilde{F}(x)$ is also a harmonic mapping. Let

$$\tilde{F}(x) = (\tilde{U}(x), \tilde{V}(x)).$$

Note that $\tilde{F}(0) = F(0) = (a, b)$. Then by the result of step 1, we have

$$\langle \tilde{U}(rN), l\rangle \leq \langle U_{a,b,r}(rN), l\rangle$$

with equality if and only if $\tilde{F}(x) = F_{a,b,r}(x)$. Note that $\tilde{U}(rN) = U(rNA^{-1}) = U(r\omega)$ and $\tilde{F}(x) = F(xA^{-1})$. Thus

$$\langle U(r\omega), l\rangle \leq \langle U_{a,b,r}(rN), l\rangle$$

with equality if and only if $F(xA^{-1}) = F_{a,b,r}(x)$. It is just that $\langle U(r\omega), l\rangle \leq \langle U_{a,b,r}(rN), l\rangle$ with equality if and only if $F(x) = F_{a,b,r}(xA)$.

Step 3: We will show that $\langle U(x), l\rangle < \langle U_{a,b,r}(rN), l\rangle$ for $|x| < r$. Let

$$g(x) = \langle U(x), l\rangle \quad \text{for } x \in \mathbb{B}^n. \tag{3.27}$$

Then $g(x)$ is a real-valued harmonic function. By the result of step 2, we know that

$$g(r\omega) \leq \langle U_{a,b,r}(rN), l\rangle.$$

Then by the maximum principle, we have

$$g(x) \leq \langle U_{a,b,r}(rN), l\rangle \quad \text{for } |x| \leq r.$$

If there exists a point x_0 with $|x_0| < r$, such that $g(x_0) = \langle U_{a,b,r}(rN), l\rangle$, then

$$g(x) \equiv \langle U_{a,b,r}(rN), l\rangle \quad \text{for } |x| \leq r. \tag{3.28}$$

Then

$$g(rN) = \langle U_{a,b,r}(rN), l\rangle.$$

Since by (3.27)

$$g(rN) = \langle U(rN), l\rangle,$$

then we have

$$\langle U(rN), l\rangle = \langle U_{a,b,r}(rN), l\rangle.$$

Then by the result of step 1, we have $U(x) = U_{a,b,r}(x)$. Thus by (3.27) and (3.28), we obtain

$$\langle U_{a,b,r}(x), l\rangle \equiv \langle U_{a,b,r}(rN), l\rangle \quad \text{for } |x| \leq r.$$

However, it is impossible since $\langle U_{a,b,r}(x), l\rangle$ is not a constant for $|x| \leq r$. Therefore, for any x with $|x| < r$, we have $g(x) < \langle U_{a,b,r}(rN), l\rangle$. The proof of the theorem is complete. $\square$

Consequently, we have a corollary as follows.

Corollary 2. *Let $F(x)$ be a harmonic mapping such that $F(\mathbb{B}^n) \subset \mathbb{B}^{m+1}$ and $F(0) = (a, b)$, where $a \in \mathbb{R}^m$ and $b \in \mathbb{R}$. Let $e_0 = (1, 0, \cdots, 0) \in \mathbb{R}^{m+1}$. Then, for $0 < r < 1$ and $\omega \in S$,*

$$\langle F(r\omega), e_0\rangle \leq \langle F_{a,b,r}(rN), e_0\rangle$$

with equality at some point $r\omega$ if and only if $F(x) = F_{a,b,r}(xA)$, where A is an orthogonal matrix such that $\omega A = N$, $F_{a,b,r}$ is defined as (3.23)*. Furthermore, $\langle F(x), e_0\rangle < \langle F_{a,b,r}(rN), e_0\rangle$ for $|x| < r$.*

Generally, we have Theorem 1 in Section 1. Now we give the proof of Theorem 1.

Proof of Theorem 1. For $x \in \mathbb{B}^n$, we have

$$\langle F(x), e\rangle = F(x)e^T = F(x)(e_0 Q_e^{-1})^T = F(x)(e_0 Q_e^T)^T = F(x)Q_e e_0^T = \langle F(x)Q_e, e_0\rangle,$$

where T is the transpose symbol. Let

$$\widetilde{F}(x) = F(x)Q_e, \qquad x \in \mathbb{B}^n.$$

Then $\widetilde{F}(x)$ is a harmonic mapping by [3], and $\widetilde{F}(\mathbb{B}^n) \subset \mathbb{B}^{m+1}$, $\widetilde{F}(0) = F(0)Q_e = (a, b)Q_e$. Using Corollary 2 to $\widetilde{F}(x)$, we have for $0 < r < 1$ and $\omega \in S$,

$$\langle \widetilde{F}(r\omega), e_0\rangle \leq \langle F_{(a,b)Q_e,r}(rN), e_0\rangle$$

with equality at some point $r\omega$ if and only if $\widetilde{F}(x) = F_{(a,b)Q_e,r}(xA)$, where A is an orthogonal matrix such that $r\omega A = rN$. Furthermore, $\langle \widetilde{F}(x), e_0\rangle < \langle F_{(a,b)Q_e,r}(rN), e_0\rangle$ for $|x| < r$. Note that for $x \in \mathbb{B}^n$, $\widetilde{F}(x) = F(x)Q_e$,

$$\langle \widetilde{F}(x), e_0\rangle = \langle F(x)Q_e, e_0\rangle = \langle F(x), e\rangle$$

and

$$\langle \widetilde{F}(r\omega), e_0\rangle = \langle F(r\omega), e\rangle.$$

Then the theorem is proved. □

From Theorem 1, we obtain Corollary 1 in Section 1. Now we give the proof of Corollary 1.

Proof of Corollary 1. We will prove the corollary by three steps.

Step 1: We claim that for $0 < r < 1$,

$$F_{0,0,r}(x) = (U(x), 0, \cdots, 0), \tag{3.29}$$

where U is the Poisson integral of the function that equals 1 on S^+ and -1 on S^-.

By Theorem 3, (3.11), (2.3) and Lemma 2, we have that

$$u_{0,0,r}(\omega) = \begin{cases} (1, 0, \cdots, 0), & \omega \in S^+; \\ (-1, 0, \cdots, 0), & \omega \in S^-. \end{cases}$$

Then by (3.20), (3.19) and (3.21), we obtain that

$$U_{0,0,r}(x) = (U(x), 0, \cdots, 0) \text{ and } V_{0,0,r}(x) \equiv 0.$$

Thus

$$F_{0,0,r}(x) = (U_{0,0,r}(x), V_{0,0,r}(x)) = (U(x), 0, \cdots, 0).$$

The claim is proved.

Step 2: For any $x \in \mathbb{B}^n$, let $|x| = r, x = r\omega$. Since $F(0) = 0$, by Theorem 1, we have that for $e_0 = (1, 0, \cdots, 0) \in \mathbb{R}^{m+1}$ and any unit vector $e \in \mathbb{R}^{m+1}$,

$$\langle F(r\omega), e\rangle \le \langle F_{0,0,r}(rN), e_0\rangle.$$

That is

$$\langle F(x), e\rangle \le \langle F_{0,0,|x|}(|x|N), e_0\rangle. \tag{3.30}$$

If $F(x) = 0$, then obviously $|F(x)| \le U(|x|N)$ since $U(|x|N) \ge 0$. If $F(x) \ne 0$, then let $e = \frac{F(x)}{|F(x)|}$ and consequently by (3.29) and (3.30), we have $|F(x)| \le U(|x|N)$.

Step 3: For some $x_0 \in \mathbb{B}^n$, let $|x_0| = r_0$. By Step 2 and Theorem 1, we have that $|F(x_0)| = U(|x_0|N)$ if and only if $F(x) = F_{0,0,r_0}(xA)Q_e^{-1}$, where A is an orthogonal matrix such that $x_0A = r_0N$, $e = \frac{F(x_0)}{|F(x_0)|}$, Q_e be an orthogonal matrix such that $eQ_e = e_0$, Q_e^{-1} is the inverse matrix of Q_e. By (3.29),

$$F_{0,0,r_0}(xA) = (U(xA), 0, \cdots, 0).$$

Note that

$$(U(xA), 0, \cdots, 0) = (U(xA), 0, \cdots, 0)e_0^T e_0,$$

where T is the transpose symbol. Then

$$F(x) = (U(xA), 0, \cdots, 0)Q_e^{-1} = ((U(xA), 0, \cdots, 0)e_0^T)(e_0Q_e^{-1}) = U(xA)e.$$

The corollary is proved. □

4. The proofs of Lemma 1 and Lemma 2

For the proofs of Lemma 1 and Lemma 2, we need the following two lemmas.

Lemma 4. *Let the determinant*

$$Q_n = \begin{vmatrix} b+a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & b+a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b+a_{nn}, \end{vmatrix} \qquad n \geq 2,$$

where $a_{ij} = -c_i c_j$ for $i \neq 1$ or $j \neq 1$. Then

(4.1)

$$Q_n = b^n + b^{n-1}(a_{11}+a_{22}+\cdots+a_{nn}) + b^{n-2}\left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \right).$$

Proof. Let

$$A_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

By the definition of the determinant Q_n, we have

$$\begin{aligned} Q_n = b^n &+ b^{n-1}(\text{the sum of all the level 1 principal minor of } A_n) \\ &+ b^{n-2}(\text{the sum of all the level 2 principal minor of } A_n) + \cdots \\ &+ b(\text{the sum of all the level n-1 principal minor of } A_n) + A_n. \end{aligned} \tag{4.2}$$

When $n = 2$, (4.1) obviously holds.

When $n \geq 3$, we will prove that for integer $k \geq 3$, the value of any level k principal minor of A_n is 0. Let P_k is a level k principal minor of A_n with $k \geq 3$. Denote

$$P_k = \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \cdots & a_{i_2 i_k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_k i_1} & a_{i_k i_2} & \cdots & a_{i_k i_k} \end{vmatrix}.$$

Let

$$\begin{aligned} A((k-1)k|pq) =& \text{the } 2\times 2 \text{ minor of } P_k \text{ that lies on the intersection of rows } (k-1), k \\ & \text{with columns } p, q, \text{ where } 1 \leq p < q \leq k, \end{aligned}$$

and

$$\begin{aligned} M((k-1)k|pq) =& \text{the } (n-2)\times(n-2) \text{ minor obtained by deleting rows } (k-1), k \\ & \text{and columns } p, q \text{ from } P_k, \text{ where } 1 \leq p < q \leq k. \end{aligned}$$

The cofactor of $A((k-1)k|pq)$ is defined to be the signed minor

$$\tilde{A}((k-1)k|pq) = (-i)^{((k-1)+k+p+q)} M((k-1)k|pq).$$

Then using Laplace's expansion to evaluate P_k in terms of the last two rows, we have

$$P_k = \sum_{1 \leq p < q \leq k} A((k-1)k|pq)\tilde{A}((k-1)k|pq). \tag{4.3}$$

By $k-1 \geq 2$, we have for any p, q $(1 \leq p < q \leq k)$,

$$A((k-1)k|pq) = \begin{vmatrix} a_{i_{k-1}i_p} & a_{i_{k-1}i_q} \\ a_{i_k i_p} & a_{i_k i_q} \end{vmatrix} = \begin{vmatrix} -c_{i_{k-1}}c_{i_p} & -c_{i_{k-1}}c_{i_q} \\ -c_{i_k}c_{i_p} & -c_{i_k}c_{i_q} \end{vmatrix} = 0. \tag{4.4}$$

Thus $P_k = 0$ by (4.3) and (4.4).

When $n \geq 3$, it is proved above that for integer $k \geq 3$, the value of any level k principal minor of A_n is 0. Then by (4.2),

$$\begin{aligned} Q_n &= b^n + b^{n-1}(\text{the sum of all the level 1 principal minor of } A_n) \\ &\quad + b^{n-2}(\text{the sum of all the level 2 principal minor of } A_n) \\ &= b^n + b^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}) + b^{n-2} \sum_{1 \leq i < j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}. \end{aligned}$$

Note that

$$\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \begin{vmatrix} -c_i c_i & -c_i c_j \\ -c_j c_i & -c_j c_j \end{vmatrix} = 0 \qquad \text{for } 1 < i < j \leq n.$$

Thus (4.1) holds when $n \geq 3$. Then the lemma is proved. ☐

Lemma 5. *Fixed integer $k \geq 1$, let matrices*

$$A = (a_{ij})_{k \times k}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix}, \quad c = (c_1, c_2, \cdots, c_k), \quad B = \begin{pmatrix} A & b \\ c & c_{k+1} \end{pmatrix}.$$

Suppose that $Ax + b = 0$ and $det(A) \neq 0$. Then

$$cx + c_{k+1} = \frac{det(B)}{det(A)}. \tag{4.5}$$

Proof. For $1 \leq i \leq k$, let A_i be the determinant obtained by replacing column j of $det(A)$ with $-b$. For $1 \leq j \leq k+1$, let B_j be the determinant obtained by deleting row $k+1$ and column j from $det(B)$.

Using Cramer's rule to $Ax + b = 0$ we have

$$x_1 = \frac{A_1}{det(A)}, x_2 = \frac{A_2}{det(A)}, \cdots, x_k = \frac{A_k}{det(A)}.$$

Then

$$c_1x_1 + c_2x_2 + \cdots + c_kx_k + c_{k+1} = \frac{1}{det(A)}(c_1A_1 + c_2A_2 + \cdots + c_kA_k + c_{k+1}det(A)). \tag{4.6}$$

Note that

$$
\begin{aligned}
&c_1A_1+c_2A_2+\cdots+c_kA_k+c_{k+1}det(A)\\
&=c_1\begin{vmatrix} -b_1 & a_{12} & a_{13} & \cdots & a_{1k}\\ -b_2 & a_{22} & a_{23} & \cdots & a_{2k}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ -b_k & a_{k2} & a_{k3} & \cdots & a_{kk}\end{vmatrix}+c_2\begin{vmatrix} a_{11} & -b_1 & a_{13} & \cdots & a_{1k}\\ a_{21} & -b_2 & a_{23} & \cdots & a_{2k}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k1} & -b_k & a_{k3} & \cdots & a_{kk}\end{vmatrix}+\cdots\\
&\quad+c_k\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(k-1)} & -b_1\\ a_{21} & a_{22} & \cdots & a_{2(k-1)} & -b_2\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k1} & a_{k2} & \cdots & a_{k(k-1)} & -b_k\end{vmatrix}+c_{k+1}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k}\\ a_{21} & a_{22} & \cdots & a_{2k}\\ \cdots & \cdots & \cdots & \cdots\\ a_{k1} & a_{k2} & \cdots & a_{kk}\end{vmatrix}\\
&=c_1(-1)^k\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1k} & b_1\\ a_{22} & a_{23} & \cdots & a_{2k} & b_2\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k2} & a_{k3} & \cdots & a_{kk} & b_k\end{vmatrix}+c_2(-1)^{k-1}\begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1k} & b_1\\ a_{21} & a_{23} & \cdots & a_{2k} & b_2\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k1} & a_{k3} & \cdots & a_{kk} & b_k\end{vmatrix}+\cdots\\
&\quad+c_k(-1)^1\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(k-1)} & b_1\\ a_{21} & a_{22} & \cdots & a_{2(k-1)} & b_2\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k1} & a_{k2} & \cdots & a_{k(k-1)} & b_k\end{vmatrix}+c_{k+1}(-1)^0\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k}\\ a_{21} & a_{22} & \cdots & a_{2k}\\ \cdots & \cdots & \cdots & \cdots\\ a_{k1} & a_{k2} & \cdots & a_{kk}\end{vmatrix}\\
&=c_1(-1)^kB_1+c_2(-1)^{k-1}B_2+\cdots+c_k(-1)^1B_k+c_{k+1}(-1)^0B_{k+1}\\
&=c_1(-1)^{k+2}B_1+c_2(-1)^{k+3}B_2+\cdots+c_k(-1)^{2k+1}B_k+c_{k+1}(-1)^{2k+2}B_{k+1}\\
&=det(B).
\end{aligned}\tag{4.7}
$$

Thus by (4.6) and (4.7), (4.5) is proved. □

Now we give the proof of Lemma 1.

Proof of Lemma 1. We will prove Lemma 1 by six steps, where Step 2 is only for the case that $m=1$, and Step 3 - Step 5 are only for the case that $m\geq 2$.

Step 1: We give some denotation and calculation. Write

$$A_{r,\lambda,\mu}(\omega)=A(\omega)=(A_1(\omega),A_2(\omega),\cdots,A_m(\omega)),$$

$$R(r,\lambda,\mu)=(R_1(r,\lambda,\mu),R_2(r,\lambda,\mu),\cdots,R_m(r,\lambda,\mu)),$$

$$l=(l_1,\cdots,l_m),\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_m),\ \text{and}\ \ a=(a_1,a_2,\cdots,a_m).$$

For $i,j=1,2,\cdots,m$, we denote $\frac{\partial R_j(r,\lambda,\mu)}{\partial\lambda_i}=R_{ji}$, $\frac{\partial R_j(r,\lambda,\mu)}{\partial\mu}=R_{j\mu}$, $\frac{\partial I(r,\lambda,\mu)}{\partial\lambda_j}=I_j$ and $\frac{\partial I(r,\lambda,\mu)}{\partial\mu}=I_\mu$. Then a simple calculation gives

$$R_{jj}=-\frac{1}{\mu}\int_S\frac{1+|A(\omega)|^2-A_j^2(\omega)}{(1+|A(\omega)|^2)^{3/2}}d\sigma\quad\text{ for } j=1,2,\cdots,m;\tag{4.8}$$

$$R_{ji}=-\frac{1}{\mu}\int_S\frac{-A_i(\omega)A_j(\omega)}{(1+|A(\omega)|^2)^{3/2}}d\sigma\quad\text{ for } i\neq j;i,j=1,2,\cdots,m;\tag{4.9}$$

$$R_{j\mu}=-\frac{1}{\mu}\int_S\frac{A_j(\omega)}{(1+|A(\omega)|^2)^{3/2}}d\sigma\quad\text{ for } j=1,2,\cdots,m;\tag{4.10}$$

$$I_j=\frac{1}{\mu}\int_S\frac{A_j(\omega)}{(1+|A(\omega)|^2)^{3/2}}d\sigma\quad\text{ for } j=1,2,\cdots,m;\tag{4.11}$$

$$I_\mu=\frac{1}{\mu}\int_S\frac{|A(\omega)|^2}{(1+|A(\omega)|^2)^{3/2}}d\sigma.\tag{4.12}$$

It is easy to see that
(i) by (4.8), for $j = 1, 2, \cdots, m$, $R_{jj} < 0$ for any $\lambda \in \mathbb{R}^m$ and $\mu > 0$, and $R_j(r, \lambda, \mu)$ is strictly decreasing as a function of λ_j for fixed the other components of λ and μ;
(ii) by (2.1) and (2.2), for $j = 1, 2, \cdots, m$, fixing μ and the components of λ expect λ_j, $R_j(r, \lambda, \mu) \to -1$ or 1 according to $\lambda_j \to +\infty$ or $\lambda_j \to -\infty$;
(iii) by (2.1) and (2.2), $0 < I(r, \lambda, \mu) < 1$ for any $\lambda \in \mathbb{R}^m$ and $\mu > 0$.

In addition, we claim that

(4.13)
$$\frac{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1(k+1)} \\ R_{21} & R_{22} & \cdots & R_{2(k+1)} \\ \cdots & \cdots & \cdots & \cdots \\ R_{(k+1)1} & R_{(k+1)2} & \cdots & R_{(k+1)(k+1)} \end{vmatrix}}{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1k} \\ R_{21} & R_{22} & \cdots & R_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ R_{k1} & R_{k2} & \cdots & R_{kk} \end{vmatrix}} < 0 \quad \text{for integer } k \text{ with } 1 \leq k \leq m-1 \text{ when } m \geq 2;$$

and

$$\frac{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1m} & R_{1\mu} \\ R_{21} & R_{22} & \cdots & R_{2m} & R_{2\mu} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} & R_{m\mu} \\ I_1 & I_2 & \cdots & I_m & I_\mu \end{vmatrix}}{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} \end{vmatrix}} > 0 \quad \text{when } m \geq 1. \tag{4.14}$$

Now we will prove the two claims above.

For (4.8)-(4.12), let $d\tilde{\sigma} = (1/(1 + |A(\omega)|^2)^{3/2})d\sigma$, $T = \int_S d\tilde{\sigma}$, $d\xi = (1/T)d\tilde{\sigma}$, $\tilde{b} = \int_S (1 + |A(\omega)|^2)d\xi$, and for $i, j = 1, 2, \cdots, m$, $\tilde{a}_{ij} = \int_S -A_i(\omega)A_j(\omega)d\xi$, $c_j = \int_S A_j(\omega)d\xi$. Then $T > 0$, $\int_S d\xi = 1$, and

$$R_{jj} = -\frac{T}{\mu}(\tilde{b} + \tilde{a}_{jj}) \quad \text{for } j = 1, 2, \cdots, m; \tag{4.15}$$

$$R_{ji} = -\frac{T}{\mu}\tilde{a}_{ij} \quad \text{for } i \neq j; i, j = 1, 2, \cdots, m; \tag{4.16}$$

$$R_{j\mu} = -\frac{T}{\mu}c_j \quad \text{for } j = 1, 2, \cdots, m; \tag{4.17}$$

$$I_j = \frac{T}{\mu}c_j \quad \text{for } j = 1, 2, \cdots, m; \tag{4.18}$$

$$I_\mu = \frac{T}{\mu}(\tilde{b} - 1); \tag{4.19}$$

$$\tilde{b} + \tilde{a}_{11} + \tilde{a}_{22} + \cdots + \tilde{a}_{jj} \geq \tilde{b} + \tilde{a}_{11} + \tilde{a}_{22} + \cdots + \tilde{a}_{mm} = 1 \quad \text{for } j = 1, 2, \cdots, m. \tag{4.20}$$

Since $A_1(\omega)=\frac{1}{\mu}\left(\frac{1}{|rN-\omega|^n}-\lambda_1\right)$ by (2.1) and $\int_S d\xi=1$, we have

$$\begin{aligned}-\tilde{a}_{11}-c_1^2&=\int_S A_1^2(\omega)d\xi-\left(\int_S A_1(\omega)d\xi\right)^2\\&=\int_S\left[A_1(\omega)-\int_S A_1(\omega)d\xi\right]^2 d\xi>0.\end{aligned}\tag{4.21}$$

When $m\geq 2$, since $\int_S d\xi=1$ and $A_j(\omega)=-\frac{\lambda_j}{\mu}$ for $j=2,\cdots,m$ by (2.1), we have

$$\tilde{a}_{ij}=-\int_S A_i(\omega)d\xi\int_S A_j(\omega)d\xi=-c_ic_j\quad\text{ for } i\neq 1 \text{ or } j\neq 1, i,j=1,2,\cdots,m;\tag{4.22}$$

and by (4.21), we have

$$\begin{vmatrix}\tilde{a}_{11}&\tilde{a}_{1j}\\ \tilde{a}_{j1}&\tilde{a}_{jj}\end{vmatrix}=\begin{vmatrix}\tilde{a}_{11}&-c_1c_j\\ -c_jc_1&-c_jc_j\end{vmatrix}=c_j^2(-\tilde{a}_{11}-c_1^2)\geq 0\quad\text{for } j=2,\cdots,m.\tag{4.23}$$

For integer $1\leq p\leq m$, let

$$Q_p=\begin{vmatrix}\tilde{b}+\tilde{a}_{11}&\tilde{a}_{12}&\cdots&\tilde{a}_{1p}\\ \tilde{a}_{21}&\tilde{b}+\tilde{a}_{22}&\cdots&\tilde{a}_{2p}\\ \vdots&\vdots&\ddots&\vdots\\ \tilde{a}_{p1}&\tilde{a}_{p2}&\cdots&\tilde{b}+\tilde{a}_{pp}\end{vmatrix}.\tag{4.24}$$

By (4.20), we have that when $p=1$, $Q_1=\tilde{b}+\tilde{a}_{11}>0$. By (4.22) and Lemma 4, we have that when $p\geq 2$,

$$\begin{aligned}Q_p&=\tilde{b}^p+\tilde{b}^{p-1}\sum_{j=1}^p\tilde{a}_{jj}+\tilde{b}^{p-2}\sum_{j=2}^p\begin{vmatrix}\tilde{a}_{11}&\tilde{a}_{1j}\\ \tilde{a}_{j1}&\tilde{a}_{jj}\end{vmatrix}\\&=\tilde{b}^{p-1}\left(\tilde{b}+\sum_{j=1}^p\tilde{a}_{jj}\right)+\tilde{b}^{p-2}\sum_{j=2}^p\begin{vmatrix}\tilde{a}_{11}&\tilde{a}_{1j}\\ \tilde{a}_{j1}&\tilde{a}_{jj}\end{vmatrix}.\end{aligned}$$

Consequently, by (4.20), (4.23) and $\tilde{b}>0$, we obtain that when $p\geq 2$, $Q_p>0$.

Let

$$Q_{m+1}=\begin{vmatrix}\tilde{b}+\tilde{a}_{11}&\tilde{a}_{12}&\cdots&\tilde{a}_{1m}&-c_1\\ \tilde{a}_{21}&\tilde{b}+\tilde{a}_{22}&\cdots&\tilde{a}_{2m}&-c_2\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ \tilde{a}_{m1}&\tilde{a}_{m2}&\cdots&\tilde{b}+\tilde{a}_{mm}&-c_m\\ -c_1&-c_2&\cdots&-c_m&\tilde{b}-1\end{vmatrix}.\tag{4.25}$$

If $m=1$, then

$$Q_{m+1}=\begin{vmatrix}\tilde{b}+\tilde{a}_{11}&-c_1\\ -c_1&\tilde{b}-1\end{vmatrix}=\tilde{b}(\tilde{b}+\tilde{a}_{11}-1)+(-\tilde{a}_{11}-c_1^2),$$

and $Q_{m+1}>0$ since (4.20) and (4.21). If $m\geq 2$, then by (4.22), $-c_j=-c_j\times 1,-1=-1\times 1$, and Lemma 4, we have

$$\begin{aligned}Q_{m+1}&=\tilde{b}^{m+1}+\tilde{b}^m\left(\sum_{j=1}^m\tilde{a}_{jj}-1\right)+\tilde{b}^{m-1}\left(\sum_{j=2}^m\begin{vmatrix}\tilde{a}_{11}&\tilde{a}_{1j}\\ \tilde{a}_{j1}&\tilde{a}_{jj}\end{vmatrix}+\begin{vmatrix}\tilde{a}_{11}&-c_1\\ -c_1&-1\end{vmatrix}\right)\\&=\tilde{b}^m\left(\tilde{b}+\sum_{j=1}^m\tilde{a}_{jj}-1\right)+\tilde{b}^{m-1}\left(\sum_{j=2}^m\begin{vmatrix}\tilde{a}_{11}&\tilde{a}_{1j}\\ \tilde{a}_{j1}&\tilde{a}_{jj}\end{vmatrix}+\begin{vmatrix}\tilde{a}_{11}&-c_1\\ -c_1&-1\end{vmatrix}\right),\end{aligned}$$

and $Q_{m+1} > 0$ since (4.20), (4.21), (4.23) and $\tilde{b} > 0$.

By (4.15), (4.16) and (4.24), we have for integer k with $1 \le k \le m-1$ when $m \ge 2$,

$$\frac{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1(k+1)} \\ R_{21} & R_{22} & \cdots & R_{2(k+1)} \\ \cdots & \cdots & \cdots & \cdots \\ R_{(k+1)1} & R_{(k+1)2} & \cdots & R_{(k+1)(k+1)} \end{vmatrix}}{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1k} \\ R_{21} & R_{22} & \cdots & R_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ R_{k1} & R_{k2} & \cdots & R_{kk} \end{vmatrix}} = \frac{\left(-\frac{T}{\mu}\right)^{k+1} \begin{vmatrix} \tilde{b}+\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1(k+1)} \\ \tilde{a}_{21} & \tilde{b}+\tilde{a}_{22} & \cdots & \tilde{a}_{2(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{(k+1)1} & \tilde{a}_{(k+1)2} & \cdots & \tilde{b}+\tilde{a}_{(k+1)(k+1)} \end{vmatrix}}{\left(-\frac{T}{\mu}\right)^{k} \begin{vmatrix} \tilde{b}+\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1k} \\ \tilde{a}_{21} & \tilde{b}+\tilde{a}_{22} & \cdots & \tilde{a}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{k1} & \tilde{a}_{k2} & \cdots & \tilde{b}+\tilde{a}_{kk} \end{vmatrix}}$$

$$= \frac{\left(-\frac{T}{\mu}\right)^{k+1} Q_{k+1}}{\left(-\frac{T}{\mu}\right)^{k} Q_k}$$

$$= \left(-\frac{T}{\mu}\right) \frac{Q_{k+1}}{Q_k}.$$

Note that $T > 0, \mu > 0, Q_k > 0, Q_{k+1} > 0$. Then the first claim (4.13) is proved.

By (4.15)-(4.19), (4.24) and (4.25), we have when $m \ge 1$,

$$\frac{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1m} & R_{1\mu} \\ R_{21} & R_{22} & \cdots & R_{2m} & R_{2\mu} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} & R_{m\mu} \\ I_1 & I_2 & \cdots & I_m & I_\mu \end{vmatrix}}{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} \end{vmatrix}} = \frac{(-1)^m \left(\frac{T}{\mu}\right)^{m+1} \begin{vmatrix} \tilde{b}+\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1m} & -c_1 \\ \tilde{a}_{21} & \tilde{b}+\tilde{a}_{22} & \cdots & \tilde{a}_{2m} & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{b}+\tilde{a}_{mm} & -c_m \\ -c_1 & -c_2 & \cdots & -c_m & \tilde{b}-1 \end{vmatrix}}{\left(-\frac{T}{\mu}\right)^{m} \begin{vmatrix} \tilde{b}+\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1m} \\ \tilde{a}_{21} & \tilde{b}+\tilde{a}_{22} & \cdots & \tilde{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{b}+\tilde{a}_{mm} \end{vmatrix}}$$

$$= \frac{(-1)^m \left(\frac{T}{\mu}\right)^{m+1} Q_{m+1}}{\left(-\frac{T}{\mu}\right)^{m} Q_m}$$

$$= \frac{T}{\mu} \frac{Q_{m+1}}{Q_m}.$$

Note that $T > 0, \mu > 0, Q_m > 0, Q_{m+1} > 0$. Then the second claim (4.14) is proved.

Step 2: Step 2 is only for the case that $m = 1$. By (i) and (ii) in Step 1, we know that for fixed μ, $R_(r, \lambda, \mu)$ is strictly decreasing from 1 to -1 as λ increasing from $-\infty$ to $+\infty$. Then for any $-1 < a < 1$ and fixed μ, there exists a unique real number $\lambda(\mu, a)$ such that

$$R(r, \lambda, \mu)\Big|_{\lambda=\lambda(\mu,a)} = a.$$

Further, using the implicit function theorem, we have that the function $\lambda = \lambda(\mu, a)$ defined on $\{(\mu, a) : \mu > 0, -1 < a < 1\}$ is a continuous function and $\frac{\partial \lambda(\mu,a)}{\partial \mu}$ exist.

Step 3: Step 3 is only for the case that $m \ge 2$. By (i) and (ii) in Step 1, we know that for fixed $\lambda_2, \cdots, \lambda_m$ and μ, $R_1(r, \lambda, \mu)$ is strictly decreasing from 1 to -1 as λ_1 increasing from $-\infty$ to $+\infty$. Then for any $-1 < a_1 < 1$ and fixed $\lambda_2, \cdots, \lambda_m$ and μ, there exists a unique real number

$\lambda_1(\lambda_2, \cdots, \lambda_m, \mu, a_1)$ such that

$$R_1(r, \lambda, \mu)\big|_{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)} = a_1.$$

Further, using the implicit function theorem, we have that the function

$$\lambda_1 = \lambda_1(\lambda_2, \cdots, \lambda_m, \mu, a_1)$$

defined on $\{(\lambda_2, \cdots, \lambda_m, \mu, a_1) : \lambda_2 \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R}, \mu > 0, -1 < a_1 < 1\}$ is a continuous function and $\frac{\partial\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)}{\partial\lambda_2}, \cdots, \frac{\partial\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)}{\partial\lambda_m}, \frac{\partial\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)}{\partial\mu}$ exist.

Step 4: For the case that $m \geq 2$, we will prove the following result:

For an integer k with $1 \leq k \leq m-1$, if

(1) there exists a unique continuous function $\lambda_1 = \lambda_1(\lambda_2, \cdots, \lambda_m, \mu, a_1)$, which defined on $\{(\lambda_2, \cdots, \lambda_m, \mu, a_1) : \lambda_2 \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R}, \mu > 0, -1 < a_1 < 1\}$, such that

$$R_1(r, \lambda, \mu)\big|_{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)} = a_1,$$

and $\frac{\partial\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)}{\partial\lambda_2}, \cdots, \frac{\partial\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)}{\partial\lambda_m}, \frac{\partial\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)}{\partial\mu}$ exist;

(2) there exists a unique continuous function $\lambda_2 = \lambda_2(\lambda_3, \cdots, \lambda_m, \mu, a_1, a_2)$, which defined on $\{(\lambda_3, \cdots, \lambda_m, \mu, a_1, a_2) : \lambda_3 \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R}, \mu > 0, a_1 \in \mathbb{R}, a_2 \in \mathbb{R}, a_1^2 + a_2^2 < 1\}$, such that

$$R_2(r, \lambda, \mu)\Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)}} = a_2,$$

and $\frac{\partial\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)}{\partial\lambda_3}, \cdots, \frac{\partial\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)}{\partial\lambda_m}, \frac{\partial\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)}{\partial\mu}$ exist;

$\vdots$

(k) there exists a unique continuous function $\lambda_k = \lambda_k(\lambda_{k+1}, \cdots, \lambda_m, \mu, a_1, \cdots, a_k)$, which defined on $\{(\lambda_{k+1}, \cdots, \lambda_m, \mu, a_1, \cdots, a_k) : \lambda_{k+1} \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R}, \mu > 0, a_1 \in \mathbb{R}, \cdots, a_k \in \mathbb{R}, a_1^2+\cdots+a_k^2 < 1\}$, such that

$$R_k(r, \lambda, \mu)\Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)\\ \cdots\cdots \\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)}} = a_k,$$

and $\frac{\partial\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)}{\partial\lambda_{k+1}}, \cdots, \frac{\partial\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)}{\partial\lambda_m}, \frac{\partial\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)}{\partial\mu}$ exist,

then

(1) if $k \leq m-2$, then there exists a unique continuous function

$$\lambda_{k+1} = \lambda_{k+1}(\lambda_{k+2}, \cdots, \lambda_m, \mu, a_1, \cdots, a_{k+1}),$$

which defined on $\{(\lambda_{k+2}, \cdots, \lambda_m, \mu, a_1, \cdots, a_{k+1}) : \lambda_{k+2} \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R}, \mu > 0, a_1 \in \mathbb{R}, \cdots, a_{k+1} \in \mathbb{R}, a_1^2 + \cdots + a_{k+1}^2 < 1\}$, such that

$$R_{k+1}(r, \lambda, \mu)\Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)\\ \cdots\cdots \\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)\\ \lambda_{k+1}=\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})}} = a_{k+1},$$

and $\frac{\partial\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})}{\partial\lambda_{k+2}}, \cdots, \frac{\partial\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})}{\partial\lambda_m}, \frac{\partial\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})}{\partial\mu}$ exist;

(2) if $k = m-1$, then there exists a unique continuous function $\lambda_m = \lambda_m(\mu, a_1, \cdots, a_m)$, which

defined on $\{(\mu, a_1, \cdots, a_m) : \mu > 0, a_1 \in \mathbb{R}, \cdots, a_m \in \mathbb{R}, a_1^2 + \cdots + a_m^2 < 1\}$, such that

$$R_{k+1}(r, \lambda, \mu)\Big|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)\\ \cdots\cdots\\ \lambda_{m-1}=\lambda_{m-1}(\lambda_m,\mu,a_1,\cdots,a_{m-1})\\ \lambda_m=\lambda_m(\mu,a_1,\cdots,a_m)}} = a_m.$$

Now we will prove the result above. For $1 \le k \le m-1$, let

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_k^*, \lambda_{k+1}, \cdots, \lambda_m) = \lambda\Big|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)\\ \cdots\cdots\\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)}},$$

where

$$\lambda_1^* = \lambda_1\Big|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)\\ \cdots\cdots\\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)}}, \lambda_2^* = \lambda_2\Big|_{\substack{\lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,\mu,a_1,a_2)\\ \cdots\cdots\\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)}},$$
$$\cdots, \lambda_k^* = \lambda_k\Big|_{\lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)}.$$

Consider the function $R_{k+1}(r, \lambda^*, \mu)$. A simple calculation gives

(4.26)
$$\frac{\partial R_{k+1}(r,\lambda^*,\mu)}{\partial\lambda_{k+1}} = \left(R_{(k+1)1}\frac{\partial\lambda_1^*}{\partial\lambda_{k+1}} + R_{(k+1)2}\frac{\partial\lambda_2^*}{\partial\lambda_{k+1}} + \cdots + R_{(k+1)k}\frac{\partial\lambda_k^*}{\partial\lambda_{k+1}} + R_{(k+1)(k+1)}\right)\Big|_{\lambda=\lambda^*}.$$

By the condition $(1)-(k)$, we have for $j = 1, 2, \cdots, k$, $R_j(r, \lambda^*, \mu) = a_j$ and consequently $\frac{\partial R_j(r,\lambda^*,\mu)}{\partial\lambda_{k+1}} = 0$, which is

$$\left(R_{j1}\frac{\partial\lambda_1^*}{\partial\lambda_{k+1}} + R_{j2}\frac{\partial\lambda_2^*}{\partial\lambda_{k+1}} + \cdots + R_{jk}\frac{\partial\lambda_k^*}{\partial\lambda_{k+1}} + R_{j(k+1)}\right)\Big|_{\lambda=\lambda^*} = 0 \text{ for } j = 1, 2, \cdots, k. \tag{4.27}$$

By (4.27), (4.26) and Lemma 5, we have

$$\frac{\partial R_{k+1}(r,\lambda^*,\mu)}{\partial\lambda_{k+1}} = \frac{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1(k+1)} \\ R_{21} & R_{22} & \cdots & R_{2(k+1)} \\ \cdots & \cdots & \cdots & \cdots \\ R_{(k+1)1} & R_{(k+1)2} & \cdots & R_{(k+1)(k+1)} \end{vmatrix}}{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1k} \\ R_{21} & R_{22} & \cdots & R_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ R_{k1} & R_{k2} & \cdots & R_{kk} \end{vmatrix}}\Bigg|_{\lambda=\lambda^*}.$$

Then by (4.13), we obtain $\frac{\partial R_{k+1}(r,\lambda^*,\mu)}{\partial\lambda_{k+1}} < 0$, which shows that $R_{k+1}(r, \lambda^*, \mu)$ is strictly decreasing as a function of λ_{k+1}. Note that $-1 < R_{k+1}(r, \lambda^*, \mu) < 1$ and $R_{k+1}(r, \lambda^*, \mu)$ is bounded since (i) and (ii) in Step 1. Thus, $R_{k+1}(r, \lambda^*, \mu)$, as a function of λ_{k+1}, respectively has finite limit as $\lambda_{k+1} \to +\infty$ and as $\lambda_{k+1} \to -\infty$.

We claim that $R_{k+1}(r, \lambda^*, \mu) \to -\sqrt{1 - a_1^2 - \cdots - a_k^2}$ as $\lambda_{k+1} \to +\infty$, and $R_{k+1}(r, \lambda^*, \mu) \to \sqrt{1 - a_1^2 - \cdots - a_k^2}$ as $\lambda_{k+1} \to -\infty$.

As $\lambda_{k+1} \to +\infty$. Note that for $j = 1, 2, \cdots, k+1$, $\left.\frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=\lambda^*}$ is bounded since $\left|\left.\frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=\lambda^*}\right| \leq 1$. Then there exists a subsequence $(\lambda_{k+1})_p \to +\infty$ such that for $j = 1, 2, \cdots, k+1$, $\left.\frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=\lambda^*|_{\lambda_{k+1}=(\lambda_{k+1})_p}}$ has a finite limit t_j. Let $(\lambda^*)_p = \lambda^*|_{\lambda_{k+1}=(\lambda_{k+1})_p}$. Then we have

$$\lim_{p\to\infty} \left.\frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=(\lambda^*)_p} = t_j \quad \text{for } j = 1, 2, \cdots, k+1. \tag{4.28}$$

We only need to prove that $R_{k+1}(r, (\lambda^*)_p, \mu) \to -\sqrt{1 - a_1^2 - \cdots - a_k^2}$ as $p \to \infty$. Let $(A(\omega))_p = ((A_1(\omega))_p, \cdots, (A_m(\omega))_p) = A_{r,(\lambda^*)_p,\mu}(\omega)$. By (2.1) and (4.28) we obtain for $j = 1, 2, \cdots, k+1$,

$$\begin{aligned}
\lim_{p\to\infty} \frac{(A_j(\omega))_p}{\sqrt{1 + |(A(\omega))_p|^2}} &= \lim_{p\to\infty} \left.\frac{\frac{1}{\mu}\left(\frac{1}{|rN-\omega|^n} l_j - \lambda_j\right)}{\sqrt{1 + \frac{1}{\mu^2}\sum_{i=1}^m \left(\frac{1}{|rN-\omega|^n} l_i - \lambda_i\right)^2}}\right|_{\lambda=(\lambda^*)_p} \\
&= \lim_{p\to\infty} \left.\frac{\frac{1}{|rN-\omega|^n}\frac{l_j}{\lambda_{k+1}} - \frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\frac{\mu^2}{\lambda_{k+1}^2} + \sum_{i=1}^m \left(\frac{1}{|rN-\omega|^n}\frac{l_i}{\lambda_{k+1}} - \frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=(\lambda^*)_p} \\
&= \lim_{p\to\infty} \left.\frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=(\lambda^*)_p} = t_j
\end{aligned} \tag{4.29}$$

uniformly for $\omega \in S$. By the Lebesgue's dominated convergence theorem and (2.2), (4.29), we have for $j = 1, 2, \cdots, k+1$,

$$\lim_{p\to\infty} R_j(r, (\lambda^*)_p, \mu) = \lim_{p\to\infty} \int_S \frac{(A_j(\omega))_p}{\sqrt{1 + |(A(\omega))_p|^2}}\, d\sigma = \int_S \lim_{p\to\infty} \frac{(A_j(\omega))_p}{\sqrt{1 + |(A(\omega))_p|^2}}\, d\sigma = t_j. \tag{4.30}$$

Note that $R_j(r, (\lambda^*)_p, \mu) \equiv a_j$ for $j = 1, 2, \cdots, k$ by the condition (1)-(k), and $\sum_{j=1}^{k+1} t_j^2 = 1$, $t_{k+1} \leq 0$ by (4.29). Then by (4.30) we have $t_j = a_j$ for $j = 1, 2, \cdots, k$, and $t_{k+1} = -\sqrt{1 - a_1^2 - \cdots - a_k^2}$. Consequently

$$\lim_{p\to\infty} R_{k+1}(r, (\lambda^*)_p, \mu) = -\sqrt{1 - a_1^2 - \cdots - a_k^2}.$$

The first claim is proved.

Using the method of the proof of the first claim, we can prove the second claim. It is proved that $R_{k+1}(r, \lambda^*, \mu)$ is continuous and strictly decreasing from $\sqrt{1 - a_1^2 - \cdots - a_k^2}$ to $-\sqrt{1 - a_1^2 - \cdots - a_k^2}$

as λ_{k+1} increasing from $-\infty$ to $+\infty$. Thus, for any

$$-\sqrt{1-a_1^2-\cdots-a_k^2}<a_{k+1}<\sqrt{1-a_1^2-\cdots-a_k^2}$$

and $a_1^2+\cdots+a_k^2<1$, we have that
(1) if $k\le m-2$, then there exists a unique real number $\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})$ such that

$$R_{k+1}(r,\lambda^*,\mu)|_{\lambda_{k+1}=\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})}=a_{k+1};$$

further, using the implicit function theorem, we have that the function $\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})$ defined on on $\{(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1}):\lambda_{k+2}\in\mathbb{R},\cdots,$ $\lambda_m\in\mathbb{R},\mu>0,a_1\in\mathbb{R},\cdots,a_{k+1}\in\mathbb{R},a_1^2+\cdots+a_{k+1}^2<1\}$ is a continuous function, and $\frac{\partial\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})}{\partial\lambda_{k+2}},\cdots,\frac{\partial\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})}{\partial\lambda_m},\frac{\partial\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,\mu,a_1,\cdots,a_{k+1})}{\partial\mu}$ exist;
(2) if $k=m-1$, then there exists a unique real number $\lambda_m(\mu,a_1,\cdots,a_m)$ such that

$$R_{k+1}(r,\lambda^*,\mu)|_{\lambda_m=\lambda_m(\mu,a_1,\cdots,a_m)}=a_m;$$

further, using the implicit function theorem, we have that the function $\lambda_m(\mu,a_1,\cdots,a_m)$ defined on on $\{(\mu,a_1,\cdots,a_m):\mu>0,a_1\in\mathbb{R},\cdots,a_m\in\mathbb{R},a_1^2+\cdots+a_m^2<1\}$ is a continuous function.

Step 5: For the case that $m\ge 2$, by Step 3 and Step 4 we have that there exists a unique continuous mapping

$$\lambda(\mu,a)=\lambda\Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,\mu,a_1)\\ \cdots\cdots\\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,\mu,a_1,\cdots,a_k)\\ \cdots\cdots\\ \lambda_m=\lambda_m(\mu,a_1,\cdots,a_m)}}$$

defined on $\{(\mu,a):\mu>0,a\in\mathbb{R}^m,a=(a_1,\cdots,a_m),|a|^2<1\}$, such that

$$R_j(r,\lambda(\mu,a),\mu)=a_j\quad\text{for } j=1,2,\cdots,m,$$

and $\frac{\partial\lambda_1(\mu,a)}{\partial\mu},\cdots,\frac{\partial\lambda_m(\mu,a)}{\partial\mu}$ exist, where $(\lambda_1(\mu,a),\cdots,\lambda_m(\mu,a))=\lambda(\mu,a)$.

Step 6: For $m\ge 1$, by Step 2 and Step 5 we know that there exists a unique continuous mapping $\lambda(\mu,a)$ defined on $\{(\mu,a):\mu>0,a\in\mathbb{R}^m,|a|^2<1\}$, such that

$$R(r,\lambda(\mu,a),\mu)=a, \tag{4.31}$$

and $\frac{\partial\lambda_1(\mu,a)}{\partial\mu},\cdots,\frac{\partial\lambda_m(\mu,a)}{\partial\mu}$ exist, where $(\lambda_1(\mu,a),\cdots,\lambda_m(\mu,a))=\lambda(\mu,a)$.

In the following, we consider the function $I(r,\lambda_(\mu,a),\mu)$.

For a fixed $a=(a_1,\cdots,a_m)\in\mathbb{R}^m$ with $|a|^2<1$, write

$$\lambda(\mu,a)=\lambda(\mu)=(\lambda_1(\mu),\cdots,\lambda_m(\mu)).$$

Then

$$\frac{dI(r,\lambda(\mu),\mu)}{d\mu}=\left(I_1\frac{d\lambda_1(\mu)}{d\mu}+I_2\frac{d\lambda_2(\mu)}{d\mu}+\cdots+I_m\frac{d\lambda_m(\mu)}{d\mu}+I_\mu\right)\Bigg|_{\lambda=\lambda(\mu)}. \tag{4.32}$$

By (4.31), we know that

$$R_j(r,\lambda(\mu),\mu)=a_j\quad\text{for } j=1,2,\cdots,m \tag{4.33}$$

and

$$\left(R_{j1}\frac{d\lambda_1(\mu)}{d\mu}+R_{j2}\frac{d\lambda_2(\mu)}{d\mu}+\cdots+R_{jm}\frac{d\lambda_m(\mu)}{d\mu}+R_{j\mu}\right)\Bigg|_{\lambda=\lambda(\mu)}=0\quad\text{for } j=1,2,\cdots,m. \tag{4.34}$$

Then by (4.34), (4.32) and Lemma 5, we have

$$\frac{dI(r,\lambda(\mu),\mu)}{d\mu} = \left.\frac{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1m} & R_{1\mu} \\ R_{21} & R_{22} & \cdots & R_{2m} & R_{2\mu} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} & R_{m\mu} \\ I_1 & I_2 & \cdots & I_m & I_\mu \end{vmatrix}}{\begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} \end{vmatrix}}\right|_{\lambda=\lambda(\mu)}.$$

By (4.14), we have $\frac{dI(r,\lambda(\mu),\mu)}{d\mu} > 0$, which shows that $I(r,\lambda(\mu),\mu)$ is strictly increasing as a function of μ. By (iii) in Step 1, we know that $I(r,\lambda(\mu),\mu)$ respectively has finite limit as $\mu \to 0$ and as $\mu \to +\infty$.

We claim that $I(r,\lambda(\mu),\mu) \to 0$ as $\mu \to 0$, and $I(r,\lambda(\mu),\mu) \to \sqrt{1-|a|^2}$ as $\mu \to +\infty$.

As $\mu \to 0$, there exists a subsequence $\mu_k \to 0$ such that $\lambda_1(\mu_k)$ has a finite limit t or tend to ∞. We only need to prove that $I(r,\lambda(\mu_k),\mu_k) \to 0$ as $k \to \infty$. Since $I(r,\lambda(\mu_k),\mu_k) = \int_S \frac{1}{\sqrt{1+|A_{r,\lambda(\mu_k),\mu_k}(\omega)|^2}}\, d\sigma$, we only need to prove that $|A_{r,\lambda(\mu_k),\mu_k}(\omega)| \to +\infty$ almost everywhere on S. Note that

$$|A_{r,\lambda(\mu_k),\mu_k}(\omega)| = \frac{1}{\mu_k}\left|\frac{1}{|rN-\omega|^n} l - \lambda(\mu_k)\right| \geq \frac{1}{\mu_k}\left|\frac{1}{|rN-\omega|^n} - \lambda_1(\mu_k)\right|$$

and

$$\frac{1}{(1+r)^n} \leq \frac{1}{|rN-\omega|^n} \leq \frac{1}{(1-r)^n}.$$

If $\lambda_1(\mu_k) \to t$ as $k \to \infty$, then $\frac{1}{|rN-\omega|^n} - \lambda_1(\mu_k)$ is bounded and $\frac{1}{|rN-\omega|^n} - \lambda_1(\mu_k) \neq 0$ almost everywhere on S. Thus $|A_{r,\lambda(\mu_k),\mu_k}(\omega)| \to +\infty$ almost everywhere on S. If $\lambda_1(\mu_k) \to \infty$ as $k \to \infty$, then it is obvious that $|A_{r,\lambda(\mu_k),\mu_k}(\omega)| \to +\infty$ uniformly for $\omega \in S$. The first claim is proved.

As $\mu \to +\infty$, $\frac{1}{\mu}\frac{1}{|rN-\omega|^n} \to 0$ uniformly for $\omega \in S$. For $j=1$ or $j=2$ or $\cdots$ or $j=m$, if there exists a subsequence $\mu_k \to +\infty$ such that $\lambda_j(\mu_k)/\mu_k \to \infty$, then $|A_{r,\lambda(\mu_k),\mu_k}(\omega)| \to +\infty$ uniformly for $\omega \in S$, and $I(r,\lambda(\mu_k),\mu_k) \to 0$, a contradiction. This shows that for $j = 1,2,\cdots,m$, $\lambda_j(\mu)/\mu$ is bounded as $\mu \to +\infty$. Thus there exists a subsequence $\mu_k \to +\infty$ such that $-\lambda_j(\mu_k)/\mu_k$ tend to a finite limit t_j for $j = 1,2,\cdots,m$. That is

$$\lim_{k\to\infty} -\lambda_j(\mu_k)/\mu_k = t_j \quad \text{for } j = 1,2,\cdots,m. \tag{4.35}$$

we only need to prove that $I(r,\lambda(\mu_k),\mu_k) \to \sqrt{1-|a|^2}$ as $k \to \infty$. Let

$$(A(\omega))_k = ((A_1(\omega))_k, \cdots, (A_m(\omega))_k) = A_{r,\lambda(\mu_k),\mu_k}(\omega).$$

By (2.1) and (4.35) we obtain for $j = 1,2,\cdots,m$,

$$\begin{aligned} \lim_{k\to\infty} \frac{(A_j(\omega))_k}{\sqrt{1+|(A(\omega))_k|^2}} &= \lim_{k\to\infty} \frac{\frac{1}{\mu}_k\left(\frac{1}{|rN-\omega|^n} l_j - \lambda_j(\mu_k)\right)}{\sqrt{1+\frac{1}{\mu_k^2}\sum_{i=1}^m \left(\frac{1}{|rN-\omega|^n} l_i - \lambda_i(\mu_k)\right)^2}} \\ &= \lim_{k\to\infty} \frac{-\frac{\lambda_j(\mu_k)}{\mu_k}}{\sqrt{1+\sum_{i=1}^m \left(\frac{\lambda_i(\mu_k)}{\mu_k}\right)^2}} = \frac{t_j}{\sqrt{1+\sum_{i=1}^m t_i^2}} \end{aligned} \tag{4.36}$$

uniformly for $\omega \in S$, and

$$
\begin{aligned}
\lim_{k\to\infty} \frac{1}{\sqrt{1+|(A(\omega))_k|^2}} &= \lim_{k\to\infty} \frac{1}{\sqrt{1+\frac{1}{\mu_k^2}\sum_{i=1}^m \left(\frac{1}{|rN-\omega|^n} l_i - \lambda_i(\mu_k)\right)^2}} \\
&= \lim_{k\to\infty} \frac{1}{\sqrt{1+\sum_{i=1}^m \left(\frac{\lambda_i(\mu_k)}{\mu_k}\right)^2}} = \frac{1}{\sqrt{1+\sum_{i=1}^m t_i^2}}
\end{aligned} \tag{4.37}
$$

uniformly for $\omega \in S$. By the Lebesgue's dominated convergence theorem and (2.2), (4.36), (4.37) we have for $j = 1, 2, \cdots, m$,

$$
\begin{aligned}
\lim_{k\to\infty} R_j(r, \lambda(\mu_k), \mu_k) &= \lim_{k\to\infty} \int_S \frac{(A_j(\omega))_k}{\sqrt{1+|(A(\omega))_k|^2}}\, d\sigma \\
&= \int_S \lim_{k\to\infty} \frac{(A_j(\omega))_k}{\sqrt{1+|(A(\omega))_k|^2}}\, d\sigma \\
&= \frac{t_j}{\sqrt{1+\sum_{i=1}^m t_i^2}},
\end{aligned} \tag{4.38}
$$

and

$$
\begin{aligned}
\lim_{k\to\infty} I(r, \lambda(\mu_k), \mu_k) &= \lim_{k\to\infty} \int_S \frac{1}{\sqrt{1+|(A(\omega))_k|^2}}\, d\sigma \\
&= \int_S \lim_{k\to\infty} \frac{1}{\sqrt{1+|(A(\omega))_k|^2}}\, d\sigma \\
&= \frac{1}{\sqrt{1+\sum_{i=1}^m t_i^2}}.
\end{aligned} \tag{4.39}
$$

Note that $R_j(r, \lambda(\mu_k), \mu_k) \equiv a_j$ for $j = 1, 2, \cdots, m$ by (4.33), and

$$
\sum_{j=1}^m \left(\frac{t_j}{\sqrt{1+\sum_{i=1}^m t_i^2}}\right)^2 + \left(\frac{1}{\sqrt{1+\sum_{i=1}^m t_i^2}}\right)^2 = 1.
$$

Then by (4.38) we obtain that $\frac{t_j}{\sqrt{1+\sum_{i=1}^m t_i^2}} = a_j$ for $j = 1, 2, \cdots, m$, and $\frac{1}{\sqrt{1+\sum_{i=1}^m t_i^2}} = \sqrt{1-|a|^2}$. Consequently by (4.39),

$$
\lim_{k\to\infty} I(r, \lambda(\mu_k), \mu_k) = \sqrt{1-|a|^2}.
$$

The second claim is proved.

It is proved that $I(r, \lambda(\mu), \mu)$ is continuous and strictly increasing from 0 to $\sqrt{1-|a|^2}$ as μ increasing from 0 to $+\infty$. Thus, for any $0 < b < \sqrt{1-|a|^2}$ and $|a| < 1$, there exists a unique real number $\mu(a, b)$ such that $I(r, \lambda(\mu(a, b)), \mu(a, b)) = b$. Further, using the implicit function theorem, we have the function $\mu(a, b)$ defined on $\{(a, b) : a \in \mathbb{R}^m, b \in \mathbb{R}, |a| < 1,\ 0 < b < \sqrt{1-|a|^2}\}$ is a continuous function.

Denote $\lambda(\mu(a, b))$ by $\lambda(r, a, b)$. Denote $\mu(a, b)$ by $\mu(r, a, b)$. We have proved that there exist a unique pair of continuous mappings $\lambda = \lambda(r, a, b)$ and $\mu = \mu(r, a, b)$ such that $R(r, \lambda(r, a, b), \mu(r, a, b)) = a$ and $I(r, \lambda(r, a, b), \mu(r, a, b)) = b$ on the upper half ball. The lemma is proved. $\square$

Now we give the proof of Lemma 2.

Proof of Lemma 2. We will prove Lemma 2 by two cases: $m = 1$ and $m \geq 2$. The case that $m = 1$ will be proved in Step 1. The case that $m \geq 2$ will be proved in Step 2 - Step 5.

Step 1: For the case that $m = 1$, we have

$$\mathcal{R}(r,\lambda) = \int_S \frac{\frac{1}{|rN-\omega|^n} - \lambda}{|\frac{1}{|rN-\omega|^n} - \lambda|}\, d\sigma = \begin{cases} 1, & \lambda \in (-\infty, \frac{1}{(1+r)^n}], \\ \int_S \frac{\frac{1}{|rN-\omega|^n} - \lambda}{|\frac{1}{|rN-\omega|^n} - \lambda|}\, d\sigma, & \lambda \in (\frac{1}{(1+r)^n}, \frac{1}{(1-r)^n}), \\ -1, & \lambda \in [\frac{1}{(1-r)^n}, +\infty). \end{cases}$$

Obviously $\mathcal{R}(r,\lambda) \equiv 1$ when $\lambda \leq \frac{1}{(1+r)^n}$, $\mathcal{R}(r,\lambda) \equiv -1$ when $\lambda \geq \frac{1}{(1-r)^n}$, and $\mathcal{R}(r,\lambda)$ is continuous and strictly decreasing from 1 to -1 as λ increasing from $\frac{1}{(1+r)^n}$ to $\frac{1}{(1-r)^n}$. Then for any $-1 < a < 1$, there exists a unique real number $\lambda(a)$ such that

$$\mathcal{R}(r,\lambda)\big|_{\lambda=\lambda(a)} = a.$$

Further, using the implicit function theorem, we have that the function $\lambda = \lambda(a)$ defined on $\{a : -1 < a < 1\}$ is a continuous function. Write $\lambda(a) = \lambda(r,a)$. Then the case that $m = 1$ is proved.

Step 2: For the case that $m \geq 2$, we give some denotation and calculation. Let

$$\mathcal{A}_{r,\lambda}(\omega) = \mathcal{A}(\omega) = (\mathcal{A}_1(\omega), \mathcal{A}_2(\omega), \cdots, \mathcal{A}_m(\omega)),$$

$$\mathcal{R}(r,\lambda) = (\mathcal{R}_1(r,\lambda), \mathcal{R}_2(r,\lambda), \cdots, \mathcal{R}_m(r,\lambda)),$$

$$l = (l_1, \cdots, l_m), \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m), \text{ and } a = (a_1, a_2, \cdots, a_m).$$

By (2.3), $|\mathcal{A}(\omega)| = \sqrt{\left(\frac{1}{|rN-\omega|^n} - \lambda_1\right)^2 + \lambda_2^2 + \cdots + \lambda_m^2}$. So if let set

$$H = \{\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m : \lambda_2 = \cdots = \lambda_m = 0\},$$

then obviously for $i, j = 1, 2, \cdots, m$, $\frac{\partial \mathcal{R}_j(r,\lambda)}{\partial \lambda_i}$ exist for $\lambda \in \mathbb{R}^m \backslash H$. We denote $\frac{\partial \mathcal{R}_j(r,\lambda)}{\partial \lambda_i} = \mathcal{R}_{ji}$ for $i, j = 1, 2, \cdots, m$. Then a simple calculation gives that for $\lambda \in \mathbb{R}^m \backslash H$,

$$\mathcal{R}_{jj} = -\int_S \frac{|\mathcal{A}(\omega)|^2 - A_j^2(\omega)}{|\mathcal{A}(\omega)|^3} d\sigma \quad \text{for } j = 1, 2, \cdots, m; \tag{4.40}$$

$$\mathcal{R}_{ji} = -\int_S \frac{-\mathcal{A}_i(\omega)\mathcal{A}_j(\omega)}{|\mathcal{A}(\omega)|^3} d\sigma \quad \text{for } i \neq j; i, j = 1, 2, \cdots, m; \tag{4.41}$$

It is easy to see that
(1) by (2.3) and (2.4), for $j = 1, 2, \cdots, m$, $\mathcal{R}_j(r,\lambda)$ is a continuous function for any $\lambda \in \mathbb{R}^m$;
(2) by (2.3) and (2.4), for $j = 1, 2, \cdots, m$, fixing the components of λ expect λ_j, $\mathcal{R}_j(r,\lambda) \to -1$ or 1 according to $\lambda_j \to +\infty$ or $\lambda_j \to -\infty$;
(3) by (2.3) and (4.40), $\mathcal{R}_{11} < 0$ for any $\lambda \in \mathbb{R}^m \backslash H$, and $\mathcal{R}_1(r,\lambda)$ is strictly decreasing as a function of λ_1 for fixed $\lambda_2, \cdots, \lambda_m$ with $\lambda_2, \cdots, \lambda_m$ are not all 0;
(4) by (2.4), for fixed $\lambda_2 = \cdots = \lambda_m = 0$,

$$\mathcal{R}_1(r,\lambda) = \int_S \frac{\frac{1}{|rN-\omega|^n} - \lambda_1}{|\frac{1}{|rN-\omega|^n} - \lambda_1|}\, d\sigma = \begin{cases} 1, & \lambda_1 \in (-\infty, \frac{1}{(1+r)^n}], \\ \int_S \frac{\frac{1}{|rN-\omega|^n} - \lambda_1}{|\frac{1}{|rN-\omega|^n} - \lambda_1|}\, d\sigma, & \lambda_1 \in (\frac{1}{(1+r)^n}, \frac{1}{(1-r)^n}), \\ -1, & \lambda_1 \in [\frac{1}{(1-r)^n}, +\infty); \end{cases}$$

(5) by (2.3) and (4.40), for $j = 2, \cdots, m$, $\mathcal{R}_{jj} < 0$ for any $\lambda \in \mathbb{R}^m$, and $\mathcal{R}_j(r,\lambda)$ is strictly decreasing as a function of λ_j for fixed the other components of λ.

In addition, for $\lambda \in \mathbb{R}^m \backslash H$, we claim that

$$\frac{\begin{vmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \cdots & \mathcal{R}_{1(k+1)} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \cdots & \mathcal{R}_{2(k+1)} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{R}_{(k+1)1} & \mathcal{R}_{(k+1)2} & \cdots & \mathcal{R}_{(k+1)(k+1)} \end{vmatrix}}{\begin{vmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \cdots & \mathcal{R}_{1k} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \cdots & \mathcal{R}_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{R}_{k1} & \mathcal{R}_{k2} & \cdots & \mathcal{R}_{kk} \end{vmatrix}} < 0 \quad \text{for integer } k \text{ with } 1 \leq k \leq m-1. \tag{4.42}$$

Now we will prove the claim above.

For (4.40) and (4.41), let $d\tilde{\sigma} = (1/|\mathcal{A}(\omega)|^3)d\sigma$, $T = \int_S d\tilde{\sigma}$, $d\xi = (1/T)d\tilde{\sigma}$, $\tilde{b} = \int_S |\mathcal{A}(\omega)|^2 d\xi$, and for $i, j = 1, 2, \cdots, m$, $\tilde{a}_{ij} = \int_S -\mathcal{A}_i(\omega)\mathcal{A}_j(\omega)d\xi$, $c_j = \int_S \mathcal{A}_j(\omega)d\xi$. Then $T > 0$, $\int_S d\xi = 1$, and

$$\mathcal{R}_{jj} = -\frac{T}{\mu}(\tilde{b} + \tilde{a}_{jj}) \quad \text{for } j = 1, 2, \cdots, m; \tag{4.43}$$

$$\mathcal{R}_{ji} = -\frac{T}{\mu}\tilde{a}_{ij} \quad \text{for } i \neq j; i, j = 1, 2, \cdots, m. \tag{4.44}$$

Since $\mathcal{A}_1(\omega) = \frac{1}{|rN-\omega|^n} - \lambda_1$ by (2.3) and $\int_S d\xi = 1$, we have

$$\begin{aligned} -\tilde{a}_{11} - c_1^2 &= \int_S A_1^2(\omega)d\xi - \left(\int_S A_1(\omega)d\xi\right)^2 \\ &= \int_S \left[A_1(\omega) - \int_S A_1(\omega)d\xi\right]^2 d\xi > 0. \end{aligned} \tag{4.45}$$

Since $\int_S d\xi = 1$ and $\mathcal{A}_j(\omega) = -\lambda_j$ for $j = 2, \cdots, m$ by (2.3), we have

$$\tilde{a}_{ij} = -\int_S \mathcal{A}_i(\omega)d\xi \int_S \mathcal{A}_j(\omega)d\xi = -c_i c_j \quad \text{for } i \neq 1 \text{ or } j \neq 1, i, j = 1, 2, \cdots, m; \tag{4.46}$$

and

$$\begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{1j} \\ \tilde{a}_{j1} & \tilde{a}_{jj} \end{vmatrix} = \begin{vmatrix} \tilde{a}_{11} & -c_1c_j \\ -c_jc_1 & -c_jc_j \end{vmatrix} = c_j^2(-\tilde{a}_{11} - c_1^2) = \lambda_j^2(-\tilde{a}_{11} - c_1^2) \text{ for } j = 2, \cdots, m. \tag{4.47}$$

For integer $1 \leq p \leq m$, let

$$Q_p = \begin{vmatrix} \tilde{b} + \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1p} \\ \tilde{a}_{21} & \tilde{b} + \tilde{a}_{22} & \cdots & \tilde{a}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{p1} & \tilde{a}_{p2} & \cdots & \tilde{b} + \tilde{a}_{pp} \end{vmatrix}. \tag{4.48}$$

Since $\lambda \in \mathbb{R}^m \backslash H$, we have that when $p = 1$, $Q_1 = \tilde{b} + \tilde{a}_{11} > 0$. By (4.46) and Lemma 4, we have that when $p \geq 2$,

$$\begin{aligned} Q_p &= \tilde{b}^p + \tilde{b}^{p-1}\sum_{j=1}^{p}\tilde{a}_{jj} + \tilde{b}^{p-2}\sum_{j=2}^{p}\begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{1j} \\ \tilde{a}_{j1} & \tilde{a}_{jj} \end{vmatrix} \\ &= \tilde{b}^{p-1}\left(\tilde{b} + \sum_{j=1}^{p}\tilde{a}_{jj}\right) + \tilde{b}^{p-2}\sum_{j=2}^{p}\begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{1j} \\ \tilde{a}_{j1} & \tilde{a}_{jj} \end{vmatrix}. \end{aligned}$$

Consequently, by (4.47), (4.45), $\tilde{b} > 0$, and $\lambda \in \mathbb{R}^m \backslash H$, we obtain that when $p \geq 2$, $Q_p > 0$.

By (4.43), (4.44) and (4.48), we have for integer k with $1 \le k \le m-1$,

$$\frac{\begin{vmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \cdots & \mathcal{R}_{1(k+1)} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \cdots & \mathcal{R}_{2(k+1)} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{R}_{(k+1)1} & \mathcal{R}_{(k+1)2} & \cdots & \mathcal{R}_{(k+1)(k+1)} \end{vmatrix}}{\begin{vmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \cdots & \mathcal{R}_{1k} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \cdots & \mathcal{R}_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{R}_{k1} & \mathcal{R}_{k2} & \cdots & \mathcal{R}_{kk} \end{vmatrix}} = \frac{(-T)^{k+1}\begin{vmatrix} \tilde{b}+\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1(k+1)} \\ \tilde{a}_{21} & \tilde{b}+\tilde{a}_{22} & \cdots & \tilde{a}_{2(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{(k+1)1} & \tilde{a}_{(k+1)2} & \cdots & \tilde{b}+\tilde{a}_{(k+1)(k+1)} \end{vmatrix}}{(-T)^{k}\begin{vmatrix} \tilde{b}+\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1k} \\ \tilde{a}_{21} & \tilde{b}+\tilde{a}_{22} & \cdots & \tilde{a}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{k1} & \tilde{a}_{k2} & \cdots & \tilde{b}+\tilde{a}_{kk} \end{vmatrix}}$$

$$= \frac{(-T)^{k+1} Q_{k+1}}{(-T)^{k} Q_k}$$

$$= (-T)\,\frac{Q_{k+1}}{Q_k}.$$

Note that $T > 0, Q_k > 0, Q_{k+1} > 0$. Then the claim (4.42) is proved.

Step 3: For the case that $m = 2$, by (1)-(3) in Step 2, we know that for fixed $\lambda_2, \cdots, \lambda_m$ with $\lambda_2, \cdots, \lambda_m$ are not all 0, $\mathcal{R}_1(r, \lambda)$ is strictly decreasing from 1 to -1 as λ_1 increasing from $-\infty$ to $+\infty$. by (4) in Step 2, we know that for fixed $\lambda_2 = \cdots = \lambda_m = 0$, $\mathcal{R}_1(r, \lambda) \equiv 1$ when $\lambda_1 \le \frac{1}{(1+r)^n}$, $\mathcal{R}_1(r, \lambda) \equiv -1$ when $\lambda_1 \ge \frac{1}{(1-r)^n}$, and $\mathcal{R}_1(r, \lambda)$ is continuous and strictly decreasing from 1 to -1 as λ_1 increasing from $\frac{1}{(1+r)^n}$ to $\frac{1}{(1-r)^n}$. Then for any $-1 < a_1 < 1$ and any fixed $\lambda_2, \cdots, \lambda_m$, there exists a unique real number $\lambda_1(\lambda_2, \cdots, \lambda_m, a_1)$ such that

$$\mathcal{R}_1(r, \lambda)\big|_{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,a_1)} = a_1.$$

Further, using the implicit function theorem, we have that the function $\lambda_1 = \lambda_1(\lambda_2, \cdots, \lambda_m, a_1)$ defined on $\{(\lambda_2, \cdots, \lambda_m, a_1) : \lambda_2 \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R}, -1 < a_1 < 1\}$ is a continuous function and $\frac{\partial \lambda_1(\lambda_2,\cdots,\lambda_m,a_1)}{\partial \lambda_2}, \cdots, \frac{\partial \lambda_1(\lambda_2,\cdots,\lambda_m,a_1)}{\partial \lambda_m}$ exist for $(\lambda_2, \cdots, \lambda_m, a_1)$ with $\lambda_2, \cdots, \lambda_m$ are not all 0.

Step 4: For the case that $m \ge 2$, we will prove the following result:

For an integer k with $1 \le k \le m-1$, if

(1) there exists a unique continuous function $\lambda_1 = \lambda_1(\lambda_2, \cdots, \lambda_m, a_1)$, which defined on $\{(\lambda_2, \cdots, \lambda_m, a_1) : \lambda_2 \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R}, -1 < a_1 < 1\}$, such that

$$\mathcal{R}_1(r, \lambda)\big|_{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,a_1)} = a_1,$$

and $\frac{\partial \lambda_1(\lambda_2,\cdots,\lambda_m,a_1)}{\partial \lambda_2}, \cdots, \frac{\partial \lambda_1(\lambda_2,\cdots,\lambda_m,a_1)}{\partial \lambda_m}$ exist for $(\lambda_2, \cdots, \lambda_m, a_1)$ with $\lambda_2, \cdots, \lambda_m$ are not all 0;

(2) there exists a unique continuous function $\lambda_2 = \lambda_2(\lambda_3, \cdots, \lambda_m, a_1, a_2)$, which defined on $\{(\lambda_3, \cdots, \lambda_m, a_1, a_2) : \lambda_3 \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R}, a_1 \in \mathbb{R}, a_2 \in \mathbb{R}, a_1^2 + a_2^2 < 1\}$, such that

$$\mathcal{R}_2(r, \lambda)\Big|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,a_1) \\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,a_1,a_2)}} = a_2,$$

and $\frac{\partial \lambda_2(\lambda_3,\cdots,\lambda_m,a_1,a_2)}{\partial \lambda_3}, \cdots, \frac{\partial \lambda_2(\lambda_3,\cdots,\lambda_m,a_1,a_2)}{\partial \lambda_m}$ exist for $(\lambda_3, \cdots, \lambda_m, a_1, a_2)$ with $\lambda_3, \cdots, \lambda_m$ are not all 0;

$\vdots$

(k) there exists a unique continuous function $\lambda_k = \lambda_k(\lambda_{k+1}, \cdots, \lambda_m, a_1, \cdots, a_k)$, which defined on $\{(\lambda_{k+1}, \cdots, \lambda_m, a_1, \cdots, a_k) : \lambda_{k+1} \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R}, a_1 \in \mathbb{R}, \cdots, a_k \in \mathbb{R}, a_1^2 + \cdots + a_k^2 < 1\}$, such

that

$$\mathcal{R}_k(r,\lambda)\Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,a_1,a_2)\\ \cdots\cdots\\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k)}} = a_k,$$

and $\frac{\partial\lambda_k(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k)}{\partial\lambda_{k+1}},\cdots,\frac{\partial\lambda_k(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k)}{\partial\lambda_m}$ exist for $(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k)$ with $\lambda_{k+1},\cdots,\lambda_m$ are not all 0,
then
(1) if $k\le m-2$, then there exists a unique continuous function $\lambda_{k+1}=\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,a_1,\cdots,a_{k+1})$, which defined on $\{(\lambda_{k+2},\cdots,\lambda_m,a_1,\cdots,a_{k+1}):\lambda_{k+2}\in\mathbb{R},\cdots,\lambda_m\in\mathbb{R},a_1\in\mathbb{R},\cdots,a_{k+1}\in\mathbb{R},a_1^2+\cdots+a_{k+1}^2<1\}$, such that

$$\mathcal{R}_{k+1}(r,\lambda)\Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,a_1,a_2)\\ \cdots\cdots\\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k)\\ \lambda_{k+1}=\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,a_1,\cdots,a_{k+1})}} = a_{k+1},$$

and $\frac{\partial\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,a_1,\cdots,a_{k+1})}{\partial\lambda_{k+2}},\cdots,\frac{\partial\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,a_1,\cdots,a_{k+1})}{\partial\lambda_m}$ exist for $(\lambda_{k+2},\cdots,\lambda_m,a_1,\cdots,a_{k+1})$ with $\lambda_{k+2},\cdots,\lambda_m$ are not all 0;
(2) if $k=m-1$, then there exists a unique continuous function $\lambda_m=\lambda_m(a_1,\cdots,a_m)$, which defined on $\{(a_1,\cdots,a_m):a_1\in\mathbb{R},\cdots,a_m\in\mathbb{R},a_1^2+\cdots+a_m^2<1\}$, such that

$$\mathcal{R}_m(r,\lambda)\Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,a_1,a_2)\\ \cdots\cdots\\ \lambda_{m-1}=\lambda_{m-1}(\lambda_m,\cdots,\lambda_m,a_1,\cdots,a_{m-1})\\ \lambda_m=\lambda_m(a_1,\cdots,a_m)}} = a_m.$$

Now we will prove the result above. For $1\le k\le m-1$, let

$$\lambda^*=(\lambda_1^*,\lambda_2^*,\cdots,\lambda_k^*,\lambda_{k+1},\cdots,\lambda_m)=\lambda\Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,a_1,a_2)\\ \cdots\cdots\\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k)}},$$

where

$$\lambda_1^*=\lambda_1\Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,a_1)\\ \lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,a_1,a_2)\\ \cdots\cdots\\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k)}},\ \lambda_2^*=\lambda_2\Bigg|_{\substack{\lambda_2=\lambda_2(\lambda_3,\cdots,\lambda_m,a_1,a_2)\\ \cdots\cdots\\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k)}},$$
$$\cdots,\lambda_k^*=\lambda_k\big|_{\lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k)}.$$

Consider the function $\mathcal{R}_{k+1}(r,\lambda^*)$. A simple calculation gives for λ^* with $\lambda_{k+1},\cdots,\lambda_m$ are not all 0,

(4.49)
$$\frac{\partial\mathcal{R}_{k+1}(r,\lambda^*)}{\partial\lambda_{k+1}}=\left(\mathcal{R}_{(k+1)1}\frac{\partial\lambda_1^*}{\partial\lambda_{k+1}}+\mathcal{R}_{(k+1)2}\frac{\partial\lambda_2^*}{\partial\lambda_{k+1}}+\cdots+\mathcal{R}_{(k+1)k}\frac{\partial\lambda_k^*}{\partial\lambda_{k+1}}+\mathcal{R}_{(k+1)(k+1)}\right)\Bigg|_{\lambda=\lambda^*}.$$

By the condition (1)-(k), we have for $j = 1, 2, \cdots, k$, $\mathcal{R}_j(r, \lambda^*) = a_j$ and consequently $\frac{\partial \mathcal{R}_j(r,\lambda^*)}{\partial \lambda_{k+1}} = 0$ for λ^* with $\lambda_{k+1}, \cdots, \lambda_m$ are not all 0, which is

$$\left(\mathcal{R}_{j1}\frac{\partial \lambda_1^*}{\partial \lambda_{k+1}} + \mathcal{R}_{j2}\frac{\partial \lambda_2^*}{\partial \lambda_{k+1}} + \cdots + \mathcal{R}_{jk}\frac{\partial \lambda_k^*}{\partial \lambda_{k+1}} + \mathcal{R}_{j(k+1)}\right)\Bigg|_{\lambda=\lambda^*} = 0 \quad \text{for } j = 1, 2, \cdots, k. \tag{4.50}$$

By (4.50), (4.49) and Lemma 5, we have

$$\frac{\partial \mathcal{R}_{k+1}(r, \lambda^*)}{\partial \lambda_{k+1}} = \left.\frac{\begin{vmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \cdots & \mathcal{R}_{1(k+1)} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \cdots & \mathcal{R}_{2(k+1)} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{R}_{(k+1)1} & \mathcal{R}_{(k+1)2} & \cdots & \mathcal{R}_{(k+1)(k+1)} \end{vmatrix}}{\begin{vmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \cdots & \mathcal{R}_{1k} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \cdots & \mathcal{R}_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{R}_{k1} & \mathcal{R}_{k2} & \cdots & \mathcal{R}_{kk} \end{vmatrix}}\right|_{\lambda=\lambda^*}$$

for λ^* with $\lambda_{k+1}, \cdots, \lambda_m$ are not all 0. Then by (4.42), we obtain $\frac{\partial \mathcal{R}_{k+1}(r,\lambda^*)}{\partial \lambda_{k+1}} < 0$ for λ^* with $\lambda_{k+1}, \cdots, \lambda_m$ are not all 0, which shows that when $\lambda_{k+1} \neq 0$, $\mathcal{R}_{k+1}(r, \lambda^*)$ is strictly decreasing as a function of λ_{k+1}. Since $\mathcal{R}_{k+1}(r, \lambda^*)$ is continuous as a function of λ_{k+1} by the condition (1)-(k) and (1) in Step 2, then for $\lambda_{k+1} \in \mathbb{R}$, $\mathcal{R}_{k+1}(r, \lambda^*)$ is strictly decreasing as a function of λ_{k+1}. Note that $-1 < \mathcal{R}_{k+1}(r, \lambda^*) < 1$ and $\mathcal{R}_{k+1}(r, \lambda^*)$ is bounded since (2) and (5) in Step 2. Thus $\mathcal{R}_{k+1}(r, \lambda^*)$, as a function of λ_{k+1}, respectively has finite limit as $\lambda_{k+1} \to +\infty$ and as $\lambda_{k+1} \to -\infty$.

We claim that $\mathcal{R}_{k+1}(r, \lambda^*) \to -\sqrt{1 - a_1^2 - \cdots - a_k^2}$ as $\lambda_{k+1} \to +\infty$, and $\mathcal{R}_{k+1}(r, \lambda^*) \to \sqrt{1 - a_1^2 - \cdots - a_k^2}$ as $\lambda_{k+1} \to -\infty$.

As $\lambda_{k+1} \to +\infty$. Note that for $j = 1, 2, \cdots, k+1$, $\left.\frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=\lambda^*}$ is bounded since $\left|\left.\frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=\lambda^*}\right| \leq 1$. Then there exists a subsequence $(\lambda_{k+1})_p \to +\infty$ such that for $j = 1, 2, \cdots, k+1$, $\left.\frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=\lambda^*|_{\lambda_{k+1}=(\lambda_{k+1})_p}}$ has a finite limit t_j. Let $(\lambda^*)_p = \lambda^*|_{\lambda_{k+1}=(\lambda_{k+1})_p}$. Then we have

$$\lim_{p\to\infty} \left.\frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}}\right|_{\lambda=(\lambda^*)_p} = t_j \quad \text{for } j = 1, 2, \cdots, k+1. \tag{4.51}$$

We only need to prove that $\mathcal{R}_{k+1}(r, (\lambda^*)_p) \to -\sqrt{1 - a_1^2 - \cdots - a_k^2}$ as $p \to \infty$. Let

$$(\mathcal{A}(\omega))_p = ((\mathcal{A}_1(\omega))_p, \cdots, (\mathcal{A}_m(\omega))_p) = \mathcal{A}_{r,(\lambda^*)_p}(\omega).$$

By (2.3) and (4.51) we obtain for $j = 1, 2, \cdots, k+1$,

$$\lim_{p\to\infty} \frac{(\mathcal{A}_j(\omega))_p}{|(\mathcal{A}(\omega))_p|} = \lim_{p\to\infty} \left. \frac{\frac{1}{|rN-\omega|^n} l_j - \lambda_j}{\sqrt{\sum_{i=1}^m \left(\frac{1}{|rN-\omega|^n} l_i - \lambda_i\right)^2}} \right|_{\lambda=(\lambda^*)_p}$$

$$(4.52) \qquad = \lim_{p\to\infty} \left. \frac{\frac{1}{|rN-\omega|^n} \frac{l_j}{\lambda_{k+1}} - \frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^m \left(\frac{1}{|rN-\omega|^n} \frac{l_i}{\lambda_{k+1}} - \frac{\lambda_i}{\lambda_{k+1}}\right)^2}} \right|_{\lambda=(\lambda^*)_p}$$

$$= \lim_{p\to\infty} \left. \frac{-\frac{\lambda_j}{\lambda_{k+1}}}{\sqrt{\sum_{i=1}^{k+1} \left(\frac{\lambda_i}{\lambda_{k+1}}\right)^2}} \right|_{\lambda=(\lambda^*)_p} = t_j$$

uniformly for $\omega \in S$. By the Lebesgue's dominated convergence theorem and (2.4), (4.52), we have for $j = 1, 2, \cdots, k+1$,

$$(4.53) \qquad \lim_{p\to\infty} \mathcal{R}_j(r, (\lambda^*)_p) = \lim_{p\to\infty} \int_S \frac{(\mathcal{A}_j(\omega))_p}{|(\mathcal{A}(\omega))_p|}\, d\sigma = \int_S \lim_{p\to\infty} \frac{(\mathcal{A}_j(\omega))_p}{|(\mathcal{A}(\omega))_p|}\, d\sigma = t_j.$$

Note that $\mathcal{R}_j(r, (\lambda^*)_p) \equiv a_j$ for $j = 1, 2, \cdots, k$ by the condition, and $\sum_{j=1}^{k+1} t_j^2 = 1$, $t_{k+1} \le 0$ by (4.52). Then by (4.53) we have $t_j = a_j$ for $j = 1, 2, \cdots, k$, and $t_{k+1} = -\sqrt{1 - a_1^2 - \cdots - a_k^2}$. Consequently

$$\lim_{p\to\infty} \mathcal{R}_{k+1}(r, (\lambda^*)_p) = -\sqrt{1 - a_1^2 - \cdots - a_k^2}.$$

The first claim is proved.

Using the method of the proof of the first claim, we can prove the second claim. It is proved that $\mathcal{R}_{k+1}(r, \lambda^*)$ is continuous and strictly decreasing from $\sqrt{1 - a_1^2 - \cdots - a_k^2}$ to $-\sqrt{1 - a_1^2 - \cdots - a_k^2}$ as λ_{k+1} increasing from $-\infty$ to $+\infty$. Therefore, for any

$$-\sqrt{1 - a_1^2 - \cdots - a_k^2} < a_{k+1} < \sqrt{1 - a_1^2 - \cdots - a_k^2}$$

with $a_1^2 - \cdots - a_k^2 < 1$, we obtain

(1) if $k \le m-2$, then there exists a unique real number $\lambda_{k+1}(\lambda_{k+2}, \cdots, \lambda_m, a_1, \cdots, a_{k+1})$ such that

$$\mathcal{R}_{k+1}(r, \lambda^*)|_{\lambda_{k+1}=\lambda_{k+1}(\lambda_{k+2},\cdots,\lambda_m,a_1,\cdots,a_{k+1})} = a_{k+1};$$

further, using the implicit function theorem, we have that the function

$$\lambda_{k+1}(\lambda_{k+2}, \cdots, \lambda_m, a_1, \cdots, a_{k+1})$$

defined on on $\{(\lambda_{k+2}, \cdots, \lambda_m, a_1, \cdots, a_{k+1}) : \lambda_{k+2} \in \mathbb{R}, \cdots, \lambda_m \in \mathbb{R} > 0, a_1 \in \mathbb{R}, \cdots, a_{k+1} \in \mathbb{R}, a_1^2 + \cdots + a_{k+1}^2 < 1\}$ is a continuous function, and

$$\frac{\partial \lambda_{k+1}(\lambda_{k+2}, \cdots, \lambda_m, a_1, \cdots, a_{k+1})}{\partial \lambda_{k+2}}, \cdots, \frac{\partial \lambda_{k+1}(\lambda_{k+2}, \cdots, \lambda_m, a_1, \cdots, a_{k+1})}{\partial \lambda_m}$$

exist;

(2) if $k = m-1$, then there exists a unique real number $\lambda_m = \lambda_m(a_1, \cdots, a_m)$ such that

$$\mathcal{R}_m(r, \lambda^*)|_{\lambda_m=\lambda_m(a_1,\cdots,a_m)} = a_m;$$

further, using the implicit function theorem, we have that the function
$\lambda_m = \lambda_m(a_1, \cdots, a_m)$ defined on $\{(a_1, \cdots, a_m) : a_1 \in \mathbb{R}, \cdots, a_m \in \mathbb{R}, a_1^2 + \cdots + a_m^2 < 1\}$ is a continuous function.

Step 5: For the case that $m \geq 2$, by Step 3 and Step 4 we have that there exists a unique continuous mapping

$$\lambda(a) = \lambda \Bigg|_{\substack{\lambda_1=\lambda_1(\lambda_2,\cdots,\lambda_m,a_1) \\ \cdots\cdots \\ \lambda_k=\lambda_k(\lambda_{k+1},\cdots,\lambda_m,a_1,\cdots,a_k) \\ \cdots\cdots \\ \lambda_m=\lambda_m(a_1,\cdots,a_m)}}$$

defined on $\{a = (a_1, \cdots, a_m) \in \mathbb{R}^m : |a| < 1\}$, such that

$$R(r, \lambda(a)) = a.$$

Write $\lambda(a) = \lambda(r, a)$. Then the case that $m \geq 2$ is proved. □

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